On the definition of factorization systems

Shaul Barkan and Jan Steinebrunner

December 15, 2025

Factorization systems play an important role in higher category theory. In this note we show that the definition usually used in the literature in terms of orthogonality is equivalent to the maybe more intuitive one, namely that "every morphism has a unique factorization up to contractible choice."

Conceptually, a factorization system on a ∞-category $\mathcal{C}$ consists of two wide subcategories $\mathcal{C}^L, \mathcal{C}^R \subseteq \mathcal{C}$ (i.e. subcategories that contain the core $\mathcal{C}^{\simeq}$), which we will generically call left and right morphisms, such that every morphism $x \to z$ in $\mathcal{C}$ has a unique factorization $f^R \circ f^L \colon x \to y \to z$ with $f^L \colon x \to y \in \mathcal{C}^L$ a left morphism and $f^R \colon y \to z \in \mathcal{C}^R$ a right morphism. A useful example to keep in mind is the category of sets $\mathcal{C} = \mathrm{Set}$ where there is a factorization system given by surjective and injective maps and every map $f \colon A \to B$ has the unique factorization $A \twoheadrightarrow \mathrm{Im}(f) \hookrightarrow B$. Factorization systems have the additional property that left and right morphisms are orthogonal, i.e. for any commutative square

$$\begin{array}{ccc} a & \longrightarrow & x \\ {\scriptstyle f}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle g} \\ b & \longrightarrow & y \end{array}$$

with $f \in \mathcal{C}^L$ and $g \in \mathcal{C}^R$ there is a unique lift l making the diagram commute. (Or more precisely, the space of such l is contractible, see Definition 2.(b).) The definition of factorization systems due to Joyal, which we recall below, takes this orthogonality as a starting point and deduces that the space of factorizations is contractible. As we shall see below, the converse is also true. We begin with the following definition:

Definition 1. Let $\mathcal{C}$ be an ∞-category. We say that a tuple of wide subcategories $(\mathcal{C}^L, \mathcal{C}^R)$ uniquely factors $\mathcal{C}$ if for any pair of objects $x, y \in \mathcal{C}$ the composition map

$$\mathfrak{c}_{x,y} \colon \operatorname*{colim}_{z \in \mathcal{C}^{\simeq}} \mathrm{Map}_{\mathcal{C}^L}(x, z) \times \mathrm{Map}_{\mathcal{C}^R}(z, y) \to \mathrm{Map}(x, y)$$

is an equivalence.

The following definition is due to Joyal [Joy07, Definition 24.4], though instead of (a) he requires a strengthening of (b) where a morphism is if left-orthogonal to $\mathcal{C}^R$ if *and only if* it is in $\mathcal{C}^L$, and vice versa. In the form stated below it can be found in [Lur09, Definition 5.2.8.8]. It turns out that (a) is redundant as proved in [GKT18, §1.1] and [Lur22, 04QF]. Below we will show that a pair of wide subcategories is a factorization system if and only if it uniquely factors.

Definition 2 (Joyal)**.** A pair of wide subcategories $(\mathcal{C}^L, \mathcal{C}^R)$ of $\mathcal{C}$ is called a factorization system if it satisfies:

(a) The full subcategories $\mathrm{Ar}^L(\mathcal{C}), \mathrm{Ar}^R(\mathcal{C}) \subseteq \mathrm{Ar}(\mathcal{C})$, spanned by morphisms in $\mathcal{C}^L$ and $\mathcal{C}^R$ respectively, are closed under retracts.

(b) Every morphism $f\colon a \to b \in \mathcal{C}^L$ is left-orthogonal to every morphism $g\colon x \to y \in \mathcal{C}^R$, i.e. the square of mapping spaces

$$\begin{array}{ccc} \mathrm{Map}(b,x) & \xrightarrow{f^*} & \mathrm{Map}(a,x) \\ {\scriptstyle g_!}\downarrow & & \downarrow{\scriptstyle g_!} \\ \mathrm{Map}(b,y) & \xrightarrow{f^*} & \mathrm{Map}(a,y) \end{array}$$

is cartesian.

(c) Every morphism $f\colon x \to y \in \mathcal{C}$ can be written as $f = f^R \circ f^L$ with $f^L \in \mathcal{C}^L$ and $f^R \in \mathcal{C}^R$.

Composition of morphisms in $\mathcal{C}$ extends to a functor

$$\circ\colon \mathrm{Ar}(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Ar}(\mathcal{C}) \simeq \mathrm{Fun}([2],\mathcal{C}) \xrightarrow{d_1} \mathrm{Fun}([1],\mathcal{C}) \simeq \mathrm{Ar}(\mathcal{C}),$$

and we let $\mathfrak{c}\colon \mathrm{Ar}^L(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Ar}^R(\mathcal{C}) \to \mathrm{Ar}(\mathcal{C})$ denote its restriction.

Proposition 3. *For a pair of wide subcategories $(\mathcal{C}^L, \mathcal{C}^R)$ of $\mathcal{C}$ the following are equivalent:*

(1) $(\mathcal{C}^L, \mathcal{C}^R)$ uniquely factor $\mathcal{C}$.

(2) $(\mathcal{C}^L, \mathcal{C}^R)$ are a factorization system in the sense of Definition 2.

(3) $(\mathcal{C}^L, \mathcal{C}^R)$ satisfy conditions (b) and (c) in Definition 2.

(4) $\mathfrak{c}\colon \mathrm{Ar}^L(\mathcal{C}) \times_{\mathcal{C}} \mathrm{Ar}^R(\mathcal{C}) \to \mathrm{Ar}(\mathcal{C})$ is an equivalence.

(5) $\mathfrak{c}^{\simeq}\colon \mathrm{Ar}^L(\mathcal{C})^{\simeq} \times_{\mathcal{C}^{\simeq}} \mathrm{Ar}^R(\mathcal{C})^{\simeq} \to \mathrm{Ar}(\mathcal{C})^{\simeq}$ is an equivalence.

Proof. The implications (2) ⇒ (3) and (4) ⇒ (5) are immediate. The equivalence (2) ⇔ (4) is the content of [Lur09, Proposition 5.2.8.17]. A proof of (3) ⇒ (2) can be found in [GKT18, §1.1] and [Lur22, 04QF].

For the implication (5) ⇒ (1) we can write

$$\operatorname*{colim}_{z \in \mathcal{C}^{\simeq}} \mathrm{Map}^L_{\mathcal{C}}(x,z) \times \mathrm{Map}^R_{\mathcal{C}}(z,y) \simeq (\mathcal{C}^L_{x/} \times_{\mathcal{C}} \mathcal{C}^R_{/z})^{\simeq},$$

because colimits over groupoids can be computed by unstraightening [Lur09, Corollary 3.3.4.6]. This shows that in the diagram

$$\begin{array}{ccccc} \mathrm{colim}_{z\in\mathcal{C}^{\simeq}} \mathrm{Map}^L_{\mathcal{C}}(x,z) \times \mathrm{Map}^R_{\mathcal{C}}(z,y) & \xrightarrow{\mathfrak{c}_{x,y}} & \mathrm{Map}_{\mathcal{C}}(x,y) & \longrightarrow & \{(x,y)\} \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \mathrm{Ar}^L(\mathcal{C})^{\simeq} \times_{\mathcal{C}^{\simeq}} \mathrm{Ar}^R(\mathcal{C})^{\simeq} & \xrightarrow[\simeq]{\mathfrak{c}^{\simeq}} & \mathrm{Ar}(\mathcal{C})^{\simeq} & \longrightarrow & \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \end{array}$$

the composite rectangle is cartesian. The right square is cartesian by definition, so pullback pasting implies that the left square is an equivalence. Since we assumed that $\mathfrak{c}^{\simeq}$ is an equivalence, it follows that $\mathfrak{c}_{x,y}$ is an equivalence for all $x, y \in \mathcal{C}$. (The converse also holds, but we do not need it here.)

For the implication (1) ⇒ (3) assume that $\mathfrak{c}_{x,z}$ is an equivalence for all $x, z \in \mathcal{C}$. We must verify the latter two conditions of Definition 2. Condition (c) is satisfied because $\mathfrak{c}_{x,z}$ is surjective on connected components. Condition (b) says that any $f\colon x_0 \to x_1 \in \mathcal{C}^L$ is left orthogonal to any $g\colon z_0 \to z_1 \in \mathcal{C}^R$, or equivalently, that the right face of the cube

$$\begin{tikzcd}
& \operatorname*{colim}_{w\in\mathcal{C}^{\simeq}} \mathrm{Map}_{\mathcal{C}^L}(x_1,w)\times\mathrm{Map}_{\mathcal{C}^R}(w,z_1) \ar[rr, "\simeq"] \ar[dd, "f^*\times\mathrm{id}" near start] & & \mathrm{Map}_{\mathcal{C}}(x_1,z_1) \ar[dd, "f^*"] \\
\operatorname*{colim}_{w\in\mathcal{C}^{\simeq}} \mathrm{Map}_{\mathcal{C}^L}(x_1,w)\times\mathrm{Map}_{\mathcal{C}^R}(w,z_0) \ar[ur, "\mathrm{id}\times g_!"] \ar[rr, "\simeq" near start, crossing over] \ar[dd, "f^*\times\mathrm{id}"] & & \mathrm{Map}_{\mathcal{C}}(x_1,z_0) \ar[ur, "g_!"] & \\
& \operatorname*{colim}_{w\in\mathcal{C}^{\simeq}} \mathrm{Map}_{\mathcal{C}^L}(x_0,w)\times\mathrm{Map}_{\mathcal{C}^R}(w,z_1) \ar[rr, "\simeq" near start] & & \mathrm{Map}_{\mathcal{C}}(x_0,z_1) \\
\operatorname*{colim}_{w\in\mathcal{C}^{\simeq}} \mathrm{Map}_{\mathcal{C}^L}(x_0,w)\times\mathrm{Map}_{\mathcal{C}^R}(w,z_0) \ar[ur, "\mathrm{id}\times g_!"] \ar[rr, "\simeq"] & & \mathrm{Map}_{\mathcal{C}}(x_0,z_0) \ar[ur, "g_!"] \ar[from=uu, "f^*" near start, crossing over] &
\end{tikzcd}$$

is cartesian. The horizontal morphisms are given by the composition maps $\mathfrak{c}_{x_i,z_j}$ and are thus equivalences, so we can equivalently consider the left face. In the ∞-category of spaces colimits indexed by ∞-groupoids commute with weakly contractible limits, and in particular pullbacks [GHK21, Lemma 2.2.8]. Therefore, it suffices to check that for each fixed y the square

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{C}^L}(x_1,y)\times\mathrm{Map}_{\mathcal{C}^R}(y,z_0) & \xrightarrow{\mathrm{id}\times g_!} & \mathrm{Map}_{\mathcal{C}^L}(x_1,y)\times\mathrm{Map}_{\mathcal{C}^R}(y,z_1) \\
{\scriptstyle f^*\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle f^*\times\mathrm{id}} \\
\mathrm{Map}_{\mathcal{C}^L}(x_0,y)\times\mathrm{Map}_{\mathcal{C}^R}(y,z_0) & \xrightarrow{\mathrm{id}\times g_!} & \mathrm{Map}_{\mathcal{C}^L}(x_0,y)\times\mathrm{Map}_{\mathcal{C}^R}(y,z_1)
\end{array}$$

is cartesian. But this square is the product of a square where the horizontal maps are equivalences with a square where the vertical maps are equivalences, so it is indeed cartesian. □

References

[GHK21] David Gepner, Rune Haugseng, and Joachim Kock. "∞-Operads as Analytic Monads". In: *International Mathematics Research Notices* 2022.16 (2021), pp. 12516–12624.

[GKT18] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. "Decomposition spaces, incidence algebras and Möbius inversion III: the decomposition space of Möbius intervals". In: *Advances in Mathematics* 334 (2018), pp. 544–584.

[Joy07] André Joyal. *Notes on quasicategories*. 2007. URL: `https://www.math.uchicago.edu/~may/IMA/Joyal.pdf`.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Ann. of Math. Stud. 170. Princeton University Press, 2009.

[Lur22] Jacob Lurie. *Kerodon – an online resource for homotopy-coherent mathematics*. `www.kerodon.net`. 2022.