Algebraic Topology II
(without proofs)

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This is a rough representation of what was discussed in the lecture, minus the proofs and pictures. I've tried to indicate what was discussed in which lecture, but the mapping from lectures to lecture notes is not always monotone. It's also not surjective: starred subsections were not discussed in the lecture.

I'll attempt to keep this document updated, at least for the first few weeks, but I'm not making any promises.

These lecture notes are mainly based on Hatcher's book [Hat02] and Randal-Williams' lecture notes [Ran21].

Contents

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Admin

- Lecturer: Jan, TA: Pierre.
- Lectures are Monday and Friday 8-10am (= 8.15-9.55 with a 10 minute break)
- Exercise sessions Tuesday 3-5pm and Friday 10-12am. In weeks with public holidays there will be lectures in the exercise slots. (See absalon.)
- Problem sheets will be uploaded on Thursday or Friday and have to be submitted on absalon by Sunday the following week.
- The exercises are discussed in the exercise sessions and don't need to be submitted, but it's a good idea to look at them before the session.
- The grade will be 50% exam and 50% the average homework grade (disregarding the lowest homework submission).

1 Homotopy theory

1.1 Homotopy groups

Motivation. The idea of algebraic topology is to study "homotopical questions" using algebraic methods. We could ask for example:

Question 1.1.1. *How many (continuous) maps $\mathbb{CP}^2 \to \mathbb{CP}^2$ are there* up to homotopy*?*

More generally we can ask:

Question 1.1.2. *Given two "reasonable" topological spaces X and Y, what is the set*

$$[X, Y] := \{\text{continuous maps } f\colon X \to Y\}/\sim$$

where we say $f \sim g$ if the maps are homotopic, i.e. there is a continuous map $H\colon X \times [0,1] \to Y$ with $H(-,0) = f$ and $H(-,1) = g$.

Often it's easier to work with pointed spaces, i.e. tuples (X, x_0) of a space X and a preferred base-point $x_0 \in X$. In that case we write

$$[(X, x_0), (Y, y_0)] := \{\text{continuous maps } f\colon X \to Y, f(x_0) = y_0\}/\sim$$

where now a homotopy H has to respect the base-point in the sense that $H(x_0, t) = y_0$ for all t.
The case of the circle $(X, x_0) = (S^1, s_0)$ will already be familiar to you: it's the fundamental group.

$$\pi_1(Y_0, y_0) := [(S^1, s_0), (Y, y_0)].$$

The definition of homotopy groups. At the heart of algebraic topology lies the following higher dimensional generalisation of the fundamental group.

Definition 1.1.3. For $n \geq 0$, the nth homotopy group of a pointed space (X, x_0) is

$$\pi_n(X, x_0) := [(S^n, s_0), (X, x_0)]$$

Example 1.1.4. We know that $\mathrm{id}_{S^2} \colon S^2 \to S^2$ is not homotopic to the constant map because it induces a non-zero map on second homology. Therefore the second homotopy group $\pi_2(S^2, s_0) = [(S^2, s_0), (S^2, s_0)]$ has at least two elements: $[\mathrm{id}_{S^2}]$ and $[\mathrm{const}_{s_0}]$. (Note: this argument can be formalised using the "Hurewicz homomorphism" $\pi_k(X, x_0) \to \widetilde{H}_k(X, x_0)$.)

For $k \geq 1$ we can define a group structure on $\pi_k(X, x_0)$ using the pinch map $S^n \to S^n \vee S^n$, by defining

$$[f] \cdot [g] := [S^n \longrightarrow S^n \vee S^n \xrightarrow{f \vee g} X]$$

Lemma 1.1.5. *$\pi_n(X, x_0)$ is a group for all $n \geq 1$ and it is abelian for $n \geq 2$.*

To prove the lemma, its more convenient to use the following alternative perspective on homotopy groups. Let $I = [0,1]$ be the unit interval, I^n the unit cube, ∂I^n its boundary (when thought of as a subset of $\mathbb{R}^n$). Then we have

$$\pi_k(X, x_0) \cong [(I^k, \partial I^k), (X, x_0)].$$

To see this note that $I^k / \partial I^k$ is homeomorphic to S^k. In the future we will switch freely between these two perspectives.

Example 1.1.6. The values of $\pi_k(S^n)$ for $k, n \leq 10$. (See Hatcher.) While these groups are in general quite difficult to determine, there are some patterns we can observe.

Remark 1.1.7. Homotopy groups are, in general, harder to compute than homology. For example, there is no closed, simply-connected, non-contractible manifold M, for which we know all homotopy groups $\pi_k(M, m_0)$. However, they are also "stronger" than homology groups. For example, we will show Whitehead's theorem, which says that a map $f \colon X \to Y$ between CW complexes is a homotopy equivalence if and only if it is an isomorphism on all homotopy groups.

***Functoriality and the homotopy category.** Given a map of pointed spaces $\varphi \colon (X, x_0) \to (Y, y_0)$ we get an induced map:

$$\begin{aligned} \varphi_* \colon \pi_k(X, x_0) &\longrightarrow \pi_k(Y, y_0) \\ [f \colon (S^k, s_0) \to (X, x_0)] &\longmapsto [\varphi \circ f \colon (S^k, s_0) \to (X, x_0) \to (Y, y_0)] \end{aligned}$$

This map is a group homomorphism, and homotopic maps $\varphi \sim \psi$ induce the same group homomorphism $\varphi_* = \psi_*$. We can encode this by saying $\pi_k(-)$ defines a functor.

Definition 1.1.8. The pointed homotopy category HoTop_* has as objects pointed topological spaces (X, x_0). Morphisms are homotopy classes of pointed maps:

$$\mathrm{Hom}_{\mathrm{HoTop}}((X, x_0), (Y, y_0)) := [(X, x_0), (Y, y_0)]$$

This allows us to formulate the functoriality of the homotopy groups. For each $k \geq 0$ we have a functor

$$\pi_k(-)\colon \mathrm{HoTop}_* \to \mathrm{Set}_*$$

that takes a pointed space (X, x_0) to the pointed set $\pi_k(X, x_0)$ and a homotopy class of pointed maps $[\varphi]$ to the map φ_*. (Note: this is the functor "corepresented" by $(S^n, s_0) \in \mathrm{HoTop}$.) When $k \geq 1$ this functor can be promoted to a functor to the category of groups

$$\pi_k(-)\colon \mathrm{HoTop}_* \to \mathrm{Grp}$$

and when $k \geq 2$ it lands in abelian groups.

As a consequence of functoriality we get:

Corollary 1.1.9. *If $\varphi\colon (X, x_0) \to (Y, y_0)$ is a pointed homotopy equivalence, then $\varphi_*\colon \pi_k(X, x_0) \to \pi_k(Y, y_0)$ is an isomorphism for all $k \geq 0$.*

This in particular means that if X is contractible, then $\pi_k(X, x_0) \cong \pi_k(\mathrm{pt}, \mathrm{pt}) = *$.

First computations. We can now compute some homotopy groups.

Lemma 1.1.10. *If $p\colon X \to Y$ is a covering, then $p_*\colon \pi_k(X, x_0) \to \pi_k(Y, p(y))$ is an isomorphism for all $k \geq 2$.*

Example 1.1.11. Here are some coverings we can apply this lemma to:

1. $\mathbb{R} \to S^1$ is a universal covering, so $\pi_k(S^1, s_0) = 0$ for $k \geq 1$.
2. $S^1 \vee S^1$ has a contractible universal cover and so does every surface Σ_g with $g \geq 2$.
3. $S^1 \times S^2$ has as a universal cover $\mathbb{R} \times S^2$.

Generalising point 3 above, we have:

Lemma 1.1.12. *For a (possibly infinite) family of based spaces $(X_\alpha, x_0^{(\alpha)})_{\alpha \in J}$ we have*

$$\pi_k\left(\prod_{\alpha \in J} X_\alpha, x_0\right) \cong \prod_{\alpha \in J} \pi_k(X_\alpha, x_0^{(\alpha)})$$

1.2 Relative homotopy groups

The definition. When working with homology groups, there is "relative" notion, where we assign a group $H_k(X, A)$ to a pair of spaces $A \subset X$. This is useful, as it allows us to talk about excision etc. In analogy with this we now introduce the relative homotopy groups of a pointed pair of spaces $x_0 \in A \subset X$. For $k \geq 1$ we define:

$$\pi_k(X, A, x_0) := [(D^n, S^{n-1}, s_0), (X, A, x_0)]$$

In other words, this set consists of maps $f\colon D^n \to X$ satisfying $f(S^{n-1}) \subset A$ and $f(s_0) = x_0$, up to homotopies that preserve these conditions.

Note that we can recover the usual homotopy groups of (X, x_0) by letting $A := \{x_0\}$:

$$\pi_k(X, \{x_0\}, x_0) = \pi_k(X, x_0).$$

Example 1.2.1. Draw $\pi_1(\Sigma_1, D^2 \amalg D^2, x_0)$, and π_2. We see that $\pi_2(X, A, x_0)$ captures loops in A that are null homotopic in X. This will be useful later in the long exact sequence of a pair.

The relative homology groups satisfied $H_k(X, A) \cong \widetilde{H}_k(X/A)$ for CW-pairs (X, A). For relative homotopy groups this doesn't hold in general, but there is a map

$$\pi_k(X, A) \longrightarrow \pi_k(X/A)$$

and we will later prove a (hard) theorem that allows us deduce that this is an isomorphism "for small enough k".

We haven't talked about group structures on relative homotopy-"groups" yet. To do so, we need the following notation:

$$\sqcap^n := \overline{(\partial I^n) \setminus (I^{n-1} \times \{0\})} \subset I^n.$$

Using this we can rewrite the relative homotopy group as

$$\pi_k(X, A, x_0) \cong [(I^n, \partial I^n, \sqcap^n), (X, A, x_0)]$$

Lemma 1.2.2. *$\pi_k(X, A, x_0)$ is a group for $k \geq 2$ and it is abelian for $k \geq 3$.*

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The compression criterion. The following allows us to more easily detect whether an element of the relative homotopy group is trivial.

Lemma 1.2.3. *An map $f\colon (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents the 0-element in $\pi_n(X, A, x_0)$ if and only if f is homotopic, relative to S^{n-1}, to a map with image in A.*

We can also express this in terms of an exact sequence:

Corollary 1.2.4. *For any pointed pair (X, A, x_0) and $k \geq 0$ the following sequence is exact:*

$$\pi_k(A, x_0) \xrightarrow{i_*} \pi_k(X, x_0) \xrightarrow{j_*} \pi_k(X, A, x_0)$$

Here the first map is induced by $i\colon (A, x_0) \to (X, x_0)$ and the second map by $j\colon (X, \{x_0\}, x_0) \to (X, A, x_0)$.

The long exact sequence of a pair. [The LES was stated in lecture 1] We can in fact extend the short exact sequence from Corollary 1.2.4 to a long exact sequence. For this we use the boundary homomorphism

$$\partial_*\colon \pi_k(X, A, x_0) \longrightarrow \pi_{k-1}(A, x_0)$$
$$[f\colon (D^k, S^{k-1}, s_0) \to (X, A, x_0)] \longmapsto [f_{|S^{k-1}}\colon (S^{k-1}, s_0) \to (A, x_0)]$$

This makes sense for all $k \geq 1$ and it is a group homomorphism for $k \geq 2$.

Proposition 1.2.5. *For any pointed CW-pair (X, A, x_0) and $k \geq 0$ the following is a long exact sequence*

$$\ldots \xrightarrow{\partial_*} \pi_k(A, x_0) \xrightarrow{i_*} \pi_k(X, x_0) \xrightarrow{j_*} \pi_k(X, A, x_0) \xrightarrow{\partial_*} \pi_{k-1}(A, x_0) \xrightarrow{i_*} \pi_{k-1}(X, x_0) \xrightarrow{j_*} \ldots$$
$$\ldots \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial_*} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)$$

Here the first map is induced by $i\colon (A, x_0) \to (X, x_0)$ and the second map by $j\colon (X, \{x_0\}, x_0) \to (X, A, x_0)$.

Remark 1.2.6. It would make sense to define $\pi_0(X, A, x_0) \coloneqq \pi_0(X)/\pi_0(A)$.

Example 1.2.7. For a space X define the cone on X to be

$$CX \coloneqq (X \times I)/(X \times \{0\}).$$

We identify $X \times \{1\} \subset CX$ with X. The cone as the useful property of being contractible while at the same time containing X. Using the long exact sequence of the pair (CX, X) we see

$$\pi_k(CX, X, x_0) \cong \pi_{k-1}(X, x_0)$$

for $k \geq 1$. So in particular the second relative homotopy group $\pi_2(CX, X, x_0)$ can be non-abelian.

***The action of the fundamental group.** [This subsection was discussed in the exercise session.]
Suppose a space X as two base points x_0 and x_1. If they lie in the same path component, then we can relate the homotopy groups at two base points as follows. Take a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. (Abbreviated $\gamma\colon x_0 \rightsquigarrow x_1$.) We define a map

$$\gamma.(-)\colon \pi_k(X, x_0) \to \pi_k(X, x_1)$$

by $\gamma.[f\colon D^n \to X] = [f']$ where we define f' on $a \in D^n$ as:

$$f'(a) = \begin{cases} f(2a) & \text{for } |a| \leq \frac{1}{2} \\ \gamma(2|a| - 1) & \text{for } |a| \geq \frac{1}{2} \end{cases}$$

This is an isomorphism of groups.
When $x_0 = x_1$, this gives an automorphism of $\pi_k(X, x_0)$. So for every loop $[\gamma] \in \pi_1(X, x_0)$ we obtain an automorphism of $\pi_k(X, x_0)$. Homotopic loops induce the same map and in fact this assembles to a group homomorphism

$$\begin{aligned} \pi_1(X, x_0) &\longrightarrow \operatorname{Aut}(\pi_n(X, x_0)) \\ [\gamma] &\mapsto ([f] \mapsto [\gamma.f]) \end{aligned}$$

We say that the group $\pi_1(X, x_0)$ acts on $\pi_k(X, x_0)$, or that $\pi_k(X, x_0)$ is a $\pi_1(X, x_0)$-module.
In the relative case we similarly we obtain a group action

$$\pi_1(A, x_0) \longrightarrow \operatorname{Aut}(\pi_n(X, A, x_0))$$

Definition 1.2.8. We say that a space X is abelian if all of the π_1-actions are trivial. Concretely, we require that for all $x_0 \in X$, $[\gamma] \in \pi_1(X, x_0)$, $[f] \in \pi_k(X, x_0)$ we have $\gamma.[f] = [f]$.

1.3 "Recollection" on CW complexes

[I'm not assuming that you are familiar with the contents of this section, so it's not really a "recollection". This section was lectured in parallel with the following one on cells and connectivity.]

Defining CW-pairs. In much of this course we will focus only on "sufficiently nice" topological spaces. Usually this will mean that we assume the spaces to be CW complexes, or at least homotopy equivalent to CW complexes. The key idea of CW complexes is that they are built from cell attachments.

Definition 1.3.1. Let X be a topological space and $\varphi\colon S^{n-1} \to X$ a map. Then the n-cell attachment along φ is defined as

$$X \cup_\varphi e^n := (X \amalg D^n)/\sim$$

where $\sim$ is the equivalence relation that identifies $p \in S^{n-1} \subset D^n$ with $\varphi(p) \in X$.

Given a (possibly infinite) family of maps $(\varphi_\alpha\colon S^{n-1} \to X)_{\alpha \in J}$, the simultaneous n-cell attachment along the φ_α is defined as

$$X \cup_\varphi \{e^n_\alpha\}_{\alpha \in J} := (X \amalg (J \times D^n))/\sim$$

where $\sim$ is the equivalence relation that identifies $(p, \alpha) \sim \varphi_\alpha(p)$ for all $p \in S^{n-1}$ and $\alpha \in J$.

An important lemma about cell attachments is the following: [This was not mentioned in the lecture.]

Lemma 1.3.2. *If $\varphi_1, \varphi_2\colon S^{k-1} \to X$ are homotopic, then $X \cup_{\varphi_1} e^k$ and $X \cup_{\varphi_2} e^k$ are homotopy equivalent.*

Now a CW complex is a space is a space obtained by simultanously attaching n-cells of successively larger dimension. We introduce the relative notion of a CW pair, the definition of a CW complex can be recovered by setting $A = \emptyset$.

Definition 1.3.3. A CW pair is a pair of topological spaces (X, A) together with a filtration (i.e. a sequence of subspaces) $A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \ldots$ such that:

1. For all $n \geq 0$, $X^{(n)}$ is obtained from $X^{(n-1)}$ by a simultaneous attachment of n-cells.
2. X is the union of the $X^{(n)}$, and $U \subset X$ is open if and only if $U \cap X^{(n)}$ is open for all n.

We say that (X, A) is (at most) n-dimensional if $X^{(n)} = X$.

Remark 1.3.4. Note that the second part of condition 2 means that a map $f\colon X \to Y$ is continuous if and only if its restriction $f_{|X^{(n)}}\colon X^{(n)} \to Y$ is continuous for all n. This is usually called the *weak topology*. Therefore, the entire condition 2 may be reformulated as saying that X is the *colimit* of the diagram $X^{(-)}\colon (\mathbb{N}, \leq) \to \mathrm{Top}$ in the category of topological spaces.

The definition of the weak topology implies that any compact subset of a CW pair (X, A) can only intersect the interior of finitely many cells in $X \setminus A$. In particular any map $D^n \to X$ factors through a finite subcomplex of X.

The homotopy extension property. We say that a pair of topological spaces (X, A) has the homotopy extension property (HEP) if for every map out of X a homotopy on A can be extended to a homotopy on X. Concretely this means:

Definition 1.3.5. A pair (X, A) has the HEP if for every space Y, every map $f\colon X \to Y$ and every homotopy $h\colon A \times [0, 1] \to Y$ with $h_{|A \times \{0\}} = f_{|A}$ there is a homotopy $H\colon X \times [0, 1] \to Y$ extending h and f, i.e. satisfying $H_{|A \times [0,1]} = h$ and $H_{|X \times \{0\}} = f$.

In diagrams this means that any of the following solid arrow commutative diagrams, needs to admit a dashed arrow that makes the diagram commute:

$$\begin{tikzcd}
A \times \{0\} \arrow[r, hook] \arrow[d, hook] & X \times \{0\} \arrow[d, hook] \arrow[ddr, bend left] & \\
A \times [0,1] \arrow[r, hook] \arrow[drr, bend right] & X \times [0,1] \arrow[dr, dashed] & \\
& & Y
\end{tikzcd}$$

Inspecting this diagram in the "universal case" (namely, when $Y = A \times [0,1] \cup_{A\times\{0\}} X \times \{0\}$), one can show:

Lemma 1.3.6. *A pair (X, A) has the HEP if and only if the inclusion*

$$A \times [0,1] \cup_{A\times\{0\}} X \times \{0\} \hookrightarrow X \times [0,1]$$

admits a retraction.

One of the great things about CW pairs is that they have the HEP. This is not terribly difficult to show, but we won't discuss it in the course.

Lemma 1.3.7. *If (X, A) is a CW pair, then it has the HEP.*

The compression lemma. We now prove a lemma that will explain why CW pairs and relative homotopy groups play so well together. In principle the reason is that CW pairs are built by gluing cells and relative homotopy groups measure how many maps from a cell a pair receives. This "lemma" is crucial to many of the proofs that follow, so it gets to be a proposition.

Proposition 1.3.8 (Compression lemma)**.** *Let (X, A) be a CW pair and (Y, B) a pair of topological spaces. Suppose that*

- *For each $n \in \mathbb{N}$ such that $X \setminus A$ has at least one n-cell, the relative homotopy group $\pi_n(Y, B)$ is trivial.*

Then every map of pairs $f\colon (X, A) \to (Y, B)$ is homotopic, relative to A, to a map with image in B.

This is proved by inductively constructing maps $f_n\colon (X, A) \to (Y, B)$ and homotopies $h_n\colon X \times [0,1] \to Y$ with $f_{-1} = f$ and

- $f_n(X^{(n)}) \subseteq B$
- $h_n\colon f_{n-1} \sim f_n$ is a homotopy relative to $X^{(n-1)}$.

These then assemble to a map $X \times [-1,\infty) \to Y$, which can be uniquely extended to a homotopy $X \times [-1,\infty] \to Y$ whose value at -1 is f and whose value at ∞ has image in B.

Whitehead's theorem for subcomplexes. As an application of the compression lemma we obtain an important special case of Whitehead's theorem.

Theorem 1.3.9 (Whitehead theorem for subcomplexes)**.** *Let (X, A) a CW pair and assume that $\pi_k(X, A, x_0) = 0$ for all $k \geq 0$ and $x_0 \in X$. Then the inclusion $A \hookrightarrow X$ admits a deformation retraction.*

Example 1.3.10 ([Problem sheet?])**.** Note that Whitehead's theorem does not hold for arbitrary pairs of spaces. The "topologists circle" $T \subset \mathbb{R}^2$ satisfies that each map $S^n \to T$ factors through a subspace homeomorphic to I and hence is homotopic to a constant map. Therefore, if we choose a good basepoint, then $\pi_k(T, t_0) = 0$ for all $k \geq 0$. If Whitehead's theorem were true, then T would be contractible. But T is not contractible, in fact there is a map $T \to S^1$ that is not homotopic to a constant map.

1.4 Cells and connectivity

Trivial homotopy groups via transversality. We now fill in "half" of the diagram of $\pi_k(S^n)$ that we saw in the first lecture.

Lemma 1.4.1. *$\pi_k(S^n, s_0) = 0$ for all $k < n$.*

This proof required us to know that every continuous map $S^k \to S^n$ is homotopic to a smooth one. In fact, the following stronger version of smooth approximation holds:

Theorem 1.4.2. *Let $f\colon D^k \to S^n$ be a continuous map. Fix two disjoint open subsets $U, V \subset D^k$ and $\varepsilon > 0$. Then there is $g\colon D^k \to S^n$ satisfying*

- $|f(x) - g(x)| < \varepsilon$ *for all* $x \in D^k$
- $f_{|U} = g_{|U}$
- $g_{|V}$ *is smooth (alternative:* $g_{|V}$ *is piecewise linear)*

Note that for $\varepsilon < 2$ the two maps are homotopic relative to U via affine interpolation.

Above we showed that the n-sphere S^n has trivial homotopy groups below dimension n. We can expect the same to work for a CW-complex that is only built from a single 0-cell and cells of dimension $\geq n$. In general, the idea seems to be that for $k < n$ a k-cell can't map non-trivially to an n-cell. Concretely, we have the following:

Proposition 1.4.3. *Let (X, A) be a CW-pair such that all cells of $X \setminus A$ have dimension $> n$, then $\pi_n(X, A) = 0$.*

The proof of this works by first noting that a map $f\colon D^n \to X$ can only hit finitely many cells and hence reducing to the case of the single cell, i.e. where $X = A \cup_{S^{m-1}} D^m$ for some $m > n$. In this case we first smooth f around the midpoint of D^m while keeping it constant in a neighbourhood of $f^{-1}(A)$.

Example 1.4.4. As a consequence of this proposition we can see, for example, that

$$\pi_k(S^n \vee S^m) \cong \pi_k(S^n) \text{ for } k \leq m - 2.$$

Using the standard cell structure on complex projective space we also see that

$$\pi_k(\mathbb{CP}^n) \cong \pi_k(\mathbb{CP}^{n+1}) \text{ for } k \leq 2n.$$

In particular $\pi_2(S^2) = \pi_2(\mathbb{CP}^1) \cong \pi_2(\mathbb{CP}^n)$ and and $\pi_3(S^2) \to \pi_3(\mathbb{CP}^n)$ is an epimorphism. (We will later see that for $n \geq 2$ this map is $\mathbb{Z} \to 0$.)

***Cellular approximation.** [We skipped this section in the lecture, but we will state the theorem in the next lecture.] The first important consequence of the compression lemma (and of the connectivity computation of Proposition 1.4.3) is that we can homotop every map of CW complexes to one that respects the cell structure.

Definition 1.4.5. We say that a map $f: X \to Y$ is cellular if for all $n \geq 0$ it sends the n-skeleton of X to the n-skeleton of Y: $f(X^{(n)}) \subset Y^{(n)}$.

Theorem 1.4.6 (Cellular approximation)**.** *Let $f: (X, A) \to (Y, B)$ be a map of pairs of CW complexes, and $A \subseteq X$ a (possibly empty) subcomplex such that $f_{|A}: A \to Y$ is cellular. Then f is homotopic relative to A to a cellular map.*

The proof of this theorem is almost identitcal to that of the compression lemma, except that we use the fact $\pi_n(Y, Y^{(n)}) = 0$ from Proposition 1.4.3 to compress $X^{(n)} \to Y$ into $Y^{(n)}$.

Example 1.4.7. If we apply this to $\varphi: (D^k, S^{k-1}) \to (Y, Y^{(n)})$ we directly recover the fact that $(Y, Y^{(n)})$ is n-connected.

1.5 Connectivity and Whitehead's theorem

Connectivity. We saw above that the n-sphere only has trivial homotopy groups below degree n. We now introduce some terminology to capture this.

Definition 1.5.1. Let $0 \leq n \leq \infty$. We say that a space X is n-connected if it is non-empty and satisfies the following equivalent conditions:

1. For all $x \in X$ and $k \leq n$ the set $\pi_k(X, x)$ is trivial.
2. For all $0 \leq k \leq n$ every map $S^k \to X$ is homotopic to a constant map.
3. For all $0 \leq k \leq n$ every map $S^k \to X$ extends to D^n.

If a space is ∞-connected we call it weakly contractible.

By convention we say that a space is -1-connected if it is non-empty and we say that every space is -2-connected.

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Definition 1.5.2. Let $n \geq 0$. We say that a map $\varphi: X \to Y$ is n-connected if $\varphi_*: \pi_k(X, x) \to \pi_k(Y, \varphi(x))$ is a bijection for all $k < n$ and a surjection for $k = n$. If a map is ∞-connected we call it a weak equivalence.

Example 1.5.3. A map $\varphi\colon X \to Y$ is 0-connected if it hits all path components of Y. It is 1-connected, if it induces a bijection on path-components and all maps induced on fundamental groups $\varphi\colon \pi_1(X,x) \to \pi_1(Y,y)$ are surjective.

The map $X \to \mathrm{pt}$ is n-connected if and only if X is $(n-1)$-connected, and the inclusion of a point $\mathrm{pt} \to X$ is n-connected if and only if X is n-connected.

The choice of requiring surjectivity in dimension n might seem slightly odd, but it's chosen such that the following works out nicely.

Lemma 1.5.4. *Let $0 \le n \le \infty$. For a pair (X,A) the following are equivalent:*

1. *The inclusion $i\colon A \to X$ is n-connected.*
2. *For all $x \in A$ and $1 \le k \le n$ the set $\pi_k(X,A,x)$ is trivial, and moreover $\pi_0(A) \to \pi_0(X)$ is surjective.*
3. *For all $0 \le k \le n$ every map of pairs $(D^k, S^{k-1}) \to (X,A)$ is homotopic relative to S^{k-1} to a map with image in A.*

We call such pairs n-connected.

Hence Proposition 1.4.3 says that if (X,A) is a CW pair such that all cells of (X,A) are of dimension $> n$, then (X,A) is n-connected.

Whitehead's theorem. Whitehead's theorem tells us that homotopy groups *detect* homotopy equivalences between CW complexes.

Theorem 1.5.5 (Whitehead)**.** *A map between CW complexes is a weak equivalence if and only if it is a homotopy equivalence.*

The following theorem tells us that up to weak equivalence every topological space may be replaced by an essentially unique CW complex. Together with Whitehead's theorem this means that "studying topological spaces up to weak equivalence is equivalent to studying CW complexes up to homotopy equivalence":

Theorem 1.5.6 (CW approximation)**.** *For any topological space T there is a weak equivalence $\varphi\colon X \to T$ where X is a CW complex. We say that the tuple (X,φ) is a CW-approximation of T. If (X,φ) and (Y,ψ) are CW approximations of the same space T, then there is a homotopy equivalence $f\colon X \to Y$ such that $\psi \circ f$ is homotopic to φ.*

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "\varphi"'] & \simeq & Y \arrow[dl, "\psi"] \\ & T & \end{tikzcd}$$

[We will not prove this theorem now because we don't need it (yet?). It will later follow from a more general construction.]

We have already proved the Whitehead theorem in the special case of those maps $A \hookrightarrow X$ that are inclusions such that (X,A) admits the structure of a CW pair. Now we reduce the general case to this.

Maps are subspaces. To prove the Whitehead theorem for arbitrary maps we need to replace an arbitrary continuous map between CW complexes by a subcomplex inclusion. This is only possible if we enlarge the target space. To do so we will need the mapping cylinder

$$M_\varphi := (X \times [0,1] \amalg Y)/\sim$$

where $\sim$ identifies $(x,1) \sim \varphi(x)$. This mapping cylinder contains a copy of $X \cong X \times \{0\} \subset M_\varphi$ as well as a copy of Y. The inclusion of Y admits a deformation retraction, making M_φ homotopy equivalent to Y. This means we can think of M_φ as a "homotopy replacement" of Y. If (X, A) is a pair and $i\colon A \hookrightarrow X$ the inclusion, then $(M_i, A) \to (X, A)$ is a homotopy equivalence of pairs. So in this sense the mapping cylinder allows us to think of any map as an inclusion. We encode this observation as follows:

Lemma 1.5.7. *Any map of CW-complex $\varphi\colon X \to Y$ can be factored as*

$$\varphi\colon X \hookrightarrow M_\varphi \xrightarrow{\simeq} Y$$

where the first map is the inclusion of a subcomplex and the second map is a homotopy equivalence.

Remark 1.5.8. Note that a map $\varphi\colon X \to Y$ is n-connected if and only if the pair (M_φ, X) is n-connected.

This allows us to deduce the general Whitehead theorem from the compression lemma **??**.

***Relative homotopy groups of maps.** [This is just a curiosity.] Using the mapping cone we define relative homotopy groups of a map that is not necessarily a subspace inclusion.

Definition 1.5.9. For a pointed map $\varphi\colon (X, x_0) \to (Y, y_0)$ we define its relative homotopy groups as

$$\pi_k\left(X \xrightarrow{\varphi} Y\right) := \pi_k(M_\varphi, X, x_0).$$

Note that this is an abuse of notation as the definition depends of φ, which is not mentioned in the notation. Using that $\pi_k(M_\varphi, y_0) \cong \pi_k(Y, y_0)$, the long exact sequence of a pair now gives us a "long exact sequence of a map":

$$\ldots \xrightarrow{\partial_*} \pi_k(X, x_0) \xrightarrow{\varphi_*} \pi_k(Y, y_0) \xrightarrow{j_*} \pi_k\left(X \xrightarrow{\varphi} Y\right) \xrightarrow{\partial_*} \pi_{k-1}(X, x_0) \xrightarrow{\varphi_*} \pi_{k-1}(X, x_0) \xrightarrow{j_*} \ldots$$

We can now say that a map φ is n-connected if and only if $\pi_k\left(X \xrightarrow{\varphi} Y\right)$ is trivial for $0 \le k \le n$.

1.6 The Hurewicz theorem

[In this section I'm not following the usual references, but rather `https://webusers.imj-prg.fr/~julien.marche/M2/topoII.pdf` (in French) pages 3-6.]

The Hurewicz homomorphism. We can define a map from the relative homotopy group to the relative homology group:

$$\begin{aligned} \hbar\colon \pi_k(X,A,x_0) &\longrightarrow H_k(X,A) \\ [f\colon (D^k,S^{k-1},s_0)\to (X,A,x_0)] &\longmapsto f_*[D^n] \end{aligned}$$

This is defined by applying the map f induces on homology $f_*\colon H_k(D^k,S^{k-1}) \to H_k(X,A)$, to the generator of $H_k(D^k,S^{k-1}) \cong \mathbb{Z}$.

Lemma 1.6.1. *The Hurewicz homomorphism is a group homomorphism, and it is invariant under the action of the fundamental group in the sense that for $[\gamma] \in \pi_1(X,x_0)$ and $[f] \in \pi_k(X,x_0)$ we have*

$$\hbar([\gamma].[f]) = [f].$$

Example 1.6.2. In the case of the n-sphere the Hurewicz homomorphism

$$\hbar\colon \pi_n(S^n,s_0) \longrightarrow H_n(S^n,s_0) \cong \mathbb{Z}$$

is surjective. (We will see below that it is an isomorphism in this degree.)

Example 1.6.3. Using the Hurewicz homomorphism in conjunction with Lemma 1.1.10 we can show that $\pi_2(S^1 \vee S^2)$ is not finitely generated.

The goal of today's lecture is to prove the Hurewicz theorem, which allows us to compute the lowest non-trivial homotopy group of a space using homology. As a motivation we state a special case:

Theorem 1.6.4 (Absolute Hurewicz theorem)**.** *Let $n \geq 2$ and suppose (X,x_0) is $(n-1)$-connected. Then $H_k(X,x_0) = 0$ for $k < n$ and the Hurewicz homomorphism in dimension n is an isomorphism*

$$\hbar\colon \pi_n(X,x_0) \xrightarrow{\cong} H_n(X).$$

Example 1.6.5. With this we can finally determine some non-trivial homotopy groups:

1. $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$.
2. $\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$ for all $n \geq 1$.
3. $\pi_n(\bigvee_{i\in I} S^n) \cong \bigoplus_{i\in I}\mathbb{Z}$ and the lower homotopy groups are trivial.

To harness the full strength of the Hurewicz theorem we will prove a relative version that applies to an n-connected pair (X,A). This brings an additional difficulty with it as A might not be simply connected, which means that we have to watch out for the action of $\pi_1(A,x_0)$. We define the following modification of the relative homotopy group:

Definition 1.6.6. Let (X,A) be a pair such that A is path-connected. The modified homotopy group of a pair $\pi'_k(X,A)$ is defined as the quotient

$$\pi'_k(X,A) := \pi_k(X,A,x_0)/\langle [f] - \gamma.[f] : f \in \pi_k(X,A,x_0), \gamma \in \pi_1(A,x_0)\rangle$$

by the normal subgroup of $\pi_k(X,A,x_0)$ generated by the differences between elements and their orbits under the $\pi_1(A,x_0)$-action. The definition is independent of the choice of base-point $x_0 \in A$.

This modified homotopy group is defined such that the Hurewicz homomorphism factors as:

$$\hbar\colon \pi_k(X,A,x_0) \twoheadrightarrow \pi'_k(X,A) \longrightarrow H_k(X,A).$$

Remark 1.6.7. [Warning: I haven't checked this, and I didn't mention it in the lectures.] The group $\pi'_k(X,A)$ is abelian (even if $k=2$) and it can be described as the quotient

$$\pi'_k(X,A) \cong \mathbb{Z}\langle[(D^k,S^{k-1}),(X,A)]\rangle/\sim$$

of the free abelian group on the set of homotopy classes of (unbased) pairs $(D^k,S^{k-1}) \to (X,A)$, by the subgroup generated by the relation $[f\vee g]-[f]-[g]$ for any two $f,g\colon (D^k,S^{k-1}) \to (X,A)$ that satisfy $f(s_0)=g(s_0)$.

A more category theoretic description of the modified homotopy group is as follows. The group $\pi_k(X,A,-)$ is a functor from the fundamental groupoid of A to the category of groups. The modified homotopy group $\pi'_k(X,A)$ is the colimit of this functor.

Modified homology groups. We now introduce an variant of the singular chain complex of a pair (X,A), which is constructed such that it is trivial in degree n and below. We will then use the compression criterion to show that it is equivalent to singular homology, if (X,A) is n-connected.

Definition 1.6.8. For a pair (X,A) and $n\geq 0$ we let

$$C_*^{(n)}(X,A) \subset C_*(X,A)$$

be the subcomplex of the singular chain complex that contains only those k-simplices

$$\sigma\colon \Delta^k \to X$$

that satisfy that the n-skeleton $(\Delta^k)^{(n)} \subset \Delta^k$ is sent to A. We let $H_*^{(n)}(X,A)$ denote the homology of this chain complex.

In a case of an n-connected pair this computes the usual homology:

Lemma 1.6.9. *If (X,A) is n-connected, then the inclusion $C_*^{(n)}(X,A) \hookrightarrow C_*(X,A)$ is a quasi-isomorphism, i.e. it induces isomorphisms $H_k^{(n)}(X,A) \cong H_k(X,A)$ for all k.*

In particular we conclude that $H_k(X,A)=0$ for $k\leq n$. By applying this to the pair (M_φ,X) for a map $\varphi\colon X\to Y$ we conclude:

Corollary 1.6.10. *If $\varphi\colon X\to Y$ is n-connected, then $\varphi_*\colon H_k(X)\to H_k(Y)$ is an isomorphism for $k<n$. In particular, a weak equivalence induces isomorphisms on all homology groups.*

We're now ready to prove the Hurewicz theorem in its full generality:

Theorem 1.6.11 (Hurewicz)**.** *For $n\geq 2$, let (X,A) be an $(n-1)$-connected pair and assume that A is path-connected. Then $H_k(X,A)=0$ for $k<n$ and in dimension n the modified Hurewicz homomorphism*

$$\hbar\colon \pi'_n(X,A) \xrightarrow{\cong} H_n(X,A)$$

is an isomorphism.

Fun fact: this proof did not use anything fancy. We could have done it in lecture 2.

Start Lecture 4

1.7 “Recollection” on cohomology

This lecture is a crash course on cohomology at the same time an introduction to Eilenberg-Mac Lane spaces. The course syllabus seems to think that you know cohomology, but I’m aware that it wasn’t covered in any courses that you could have taken. While we don’t have time to go through these things in detail, I hope that this overview will set a good basis for when we actually want to use these things later in the course.

One reason we might care about cohomology is that it appears in the formulation of Poincaré duality:

Theorem 1.7.1 (Poincaré duality)**.** *Let M be a closed n-dimensional manifold and suppose that M is oriented. Then there are isomorphisms*

$$D\colon H^k(M;\mathbb{Z}) \xrightarrow{\cong} H_{n-k}(M;\mathbb{Z}).$$

If we work with coefficients in a field k, then we can replace cohomology by the dual of homology:

Theorem 1.7.2. *Let k be a field and M a closed n-dimensional manifold. If* $\mathrm{char}(k) \neq 2$, *assume that M is oriented. Then there are isomorphisms*

$$D\colon \mathrm{Hom}_k(H_{n-k}(M;k),k) \cong H^k(M;k) \xrightarrow{\cong} H^{n-k}(M;k).$$

However, we would like to understand Poincaré duality integrally, and we would also like to understand how to actually construct the duality isomorphisms D. This requires us to talk more about cohomology, and along the way we’ll see that cohomology is also a very useful tool in its own right.

Cellular cohomology. [Apparently cellular homology wasn’t covered in AlgTopI. Apologies for assuming familiarity with it here.] Recall the cellular chain complex of a CW-complex X. The abelian group of cellular k-chains is the free abelian group on the set of cells

$$C_k^{\mathrm{cell}}(X) \coloneqq \mathbb{Z}\langle e^k \subset X \text{ cell}\rangle$$

and the boundary operator is defined using the attaching maps of the cells.

$$\begin{aligned} \partial\colon C_k^{\mathrm{cell}}(X) &\longrightarrow C_{k-1}^{\mathrm{cell}}(X), \\ \sigma &\longmapsto \partial\sigma = \sum_{e^{k-1}\subset\partial\sigma} \pm e^{k-1} \end{aligned}$$

Cellular homology of X is the homology of the chain complex

$$H_*^{\mathrm{cell}}(X) \coloneqq H_*\left(C_*^{\mathrm{cell}}(X);\partial\right)$$

We now can define cellular cohomology. As “coefficients” we take some abelian group G (think $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{Z}/p$).

Definition 1.7.3. A G-valued cellular cochain on X is a map

$$\alpha\colon \{e^k \subset X \text{ cell}\} \to G$$

from the set of cells of X to G. The cellular cochain complex of X with coefficients in G is

$$C^k_{\text{cell}}(X;G) := \operatorname{Hom}(C_k^{\text{cell}}(X), G) = \operatorname{Map}(\{\sigma\colon \Delta^k \to X\}, G)$$

and the coboundary operator is

$$\begin{aligned} \delta\colon C^k_{\text{cell}}(X) &\longrightarrow C^{k+1}_{\text{cell}}(X), \\ \alpha &\longmapsto \alpha \circ \partial. \end{aligned}$$

This means that $(\delta\alpha)(\sigma) := \alpha(\partial\sigma)$. This results in the complex

$$C^0_{\text{cell}}(X;R) \xrightarrow{\delta} C^1_{\text{cell}}(X;R) \xrightarrow{\delta} C^2_{\text{cell}}(X;R) \xrightarrow{\delta} \dots$$

and the cellular cohomology is defined as the cohomology of this complex:

$$H^k_{\text{cell}}(X;R) := \frac{\text{cocycles}}{\text{coboundaries}} := \frac{\ker\left(\delta\colon C^k_{\text{cell}}(X;R) \to C^{k+1}_{\text{cell}}(X;R)\right)}{\operatorname{Im}\left(\delta\colon C^{k-1}_{\text{cell}}(X;R) \to C^k_{\text{cell}}(X;R)\right)}$$

Note that $\alpha\colon C_k^{\text{cell}}(X) \to G$ is a cocycle, if and only if $\alpha(\partial x) = 0$ for all $(n+1)$-cells x. We say that α vanishes on boundaries. This allows us to construct a well-defined duality pairing:

$$\begin{aligned} H_k(X;R) \otimes H^k(X;R) &\longrightarrow R \\ [c] \otimes [\alpha] &\longmapsto \alpha(c) \end{aligned}$$

where c is a cycle and α is a cocycle.

Example 1.7.4. The cellular chain complex for $\mathbb{CP}^2$ looks as follows:

$$\dots \to 0 \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z}$$

and hence its homology is $H_*(\mathbb{CP}^2) \cong \mathbb{Z}$ for $* \in \{0,2,4\}$ and 0 otherwise. The cellular cochain complex (with R-coefficients) is

$$\dots \leftarrow 0 \leftarrow 0 \leftarrow R \leftarrow 0 \leftarrow R \leftarrow 0 \leftarrow R.$$

and hence its cohomology is $H^*(\mathbb{CP}^2;R) \cong R$ for $* \in \{0,2,4\}$ and 0 otherwise. Not very interesting.

Example 1.7.5. The cellular chain complex for $\mathbb{RP}^3$ looks as follows:

$$\dots \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

and hence its homology is $H_*(\mathbb{RP}^3) \cong (\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z})$ for $* = (0,1,2,3)$ and 0 otherwise. The cellular cochain complex (with R-coefficients) is

$$\dots \to 0 \longleftarrow R \xleftarrow{0} R \xleftarrow{\cdot 2} R \xleftarrow{0} R$$

and hence its cohomology is $H^*(\mathbb{RP}^3;R) \cong (R, R^{2-\text{tor}}, R/2, R)$. This is much more interesting. Even for $R = \mathbb{Z}$ we get: $H^*(\mathbb{RP}^3;\mathbb{Z}) \cong (\mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z})$.

Example 1.7.6. The cellular chain complex for the Klein bottle K looks as follows:

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{(2,2)} \mathbb{Z} \times \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

and hence its homology is $H_*(\mathbb{RP}^3) \cong (\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}/2, 0)$ for $* = (0, 1, 2)$ and 0 otherwise. The cellular cochain complex (with $\mathbb{Z}$-coefficients) is

$$\cdots \to 0 \longleftarrow \mathbb{Z} \xleftarrow{\binom{2}{2}} \mathbb{Z} \times \mathbb{Z} \xleftarrow{0} \mathbb{Z}$$

and hence its cohomology is $H^*(\mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}/2)$.

The universal coefficient theorem. In Example 1.7.5 we observed that passing to cohomology moves the torsion groups up by one degree, but doesn't do much otherwise. This observation can be formalised in the following way:

Theorem 1.7.7 (Universal coefficient theorem)**.** *For any X there is a natural short exact sequence:*

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(X; \mathbb{Z}), R) \longrightarrow H^n(X; R) \longrightarrow \mathrm{Hom}(H_n(X; \mathbb{Z}), R) \longrightarrow 0$$

This sequence always splits, but the splitting is not natural.

Remark 1.7.8. For $R = \mathbb{Z}$, the Ext^1 term can be computed as follows. Suppose that M is a finitely generated abelian group. Then we can write $M \cong \mathbb{Z}^r \oplus T$ where T is a finite abelian group. In this case the Ext^1 group is

$$\mathrm{Ext}^1(\mathbb{Z}^r \oplus T, \mathbb{Z}) \cong T.$$

(Warning: this isomorphism is not canonical. To get a canonical isomorphism, we should write $T^\vee = \mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z})$ instead of T. But this is always abstractly isomorphic to T.)

Remark 1.7.9. If X is a finite CW complex, then we can write $H_n(X; \mathbb{Z}) \cong \mathbb{Z}^{r_n} \oplus T_n$ with T_n finite, and the above remark together with the (split) short exact sequence from the UCT gives us a *non-canonical* isomorphism

$$H^n(X; \mathbb{Z}) \cong \mathbb{Z}^{r_n} \oplus T_{n-1}.$$

Example 1.7.10. For a closed oriented n-manifold wrote $H_k(X; \mathbb{Z}) \cong \mathbb{Z}^{b_k} \oplus T_k$ with T_k finite. Then Poincaré duality together with the UCT imply:

$$b_k = b_{n-k} \qquad \text{and} \qquad T_k \cong T_{n-k-1}$$

for all $0 \leq k \leq n$. In particular, $H_{n-1}(X; \mathbb{Z})$ must be torsion free.

Singular cohomology. Recall the singular chain complex of a topological space X. A singular chain is an abstract linear combination of continuous maps $\sigma\colon \Delta^k \to X$. The abelian group of singular k-chains is the free abelian group on the set of singular simplices

$$C_k^{\mathrm{sing}}(X) := \mathbb{Z}\langle \sigma\colon \Delta^k \to X \text{ continuous} \rangle$$

and the boundary operator is defined as the alternating sum over the faces:

$$\begin{aligned}\partial\colon C_k^{\mathrm{sing}}(X) &\longrightarrow C_{k-1}^{\mathrm{sing}}(X),\\ \sigma &\longmapsto \sum_{i=0}^{k}(-1)^i d_i\sigma\end{aligned}$$

Just like in the cellular case, singular cohomology is defined as the G-dual of singular homology. We now can define singular cohomology.

Definition 1.7.11. A G-valued singluar cochain on X is a map

$$\alpha\colon \{\sigma\colon \Delta^k \to X\} \to G$$

from the set of singular simplices in X to G. The singular cochain complex of X with coefficients in G is

$$C^k_{\mathrm{sing}}(X;G) := \mathrm{Hom}(C_k^{\mathrm{sing}}(X), G) = \mathrm{Map}(\{\sigma\colon \Delta^k \to X\}, G)$$

and the coboundary operator is

$$\begin{aligned}\delta\colon C^k_{\mathrm{sing}}(X) &\longrightarrow C^{k+1}_{\mathrm{sing}}(X),\\ \alpha &\longmapsto \alpha\circ\partial.\end{aligned}$$

This means that $(\delta\alpha)(\sigma) := \alpha(\partial\sigma)$. This results in the complex

$$C^0_{\mathrm{sing}}(X;G) \xrightarrow{\delta} C^1_{\mathrm{sing}}(X;G) \xrightarrow{\delta} C^2_{\mathrm{sing}}(X;G) \xrightarrow{\delta} \dots$$

and the singular cohomology is defined as the cohomology of this complex:

$$H^k_{\mathrm{sing}}(X;G) := \frac{\text{cocycles}}{\text{coboundaries}} := \frac{\ker\left(\delta\colon C^k_{\mathrm{sing}}(X;G) \to C^{k+1}_{\mathrm{sing}}(X;G)\right)}{\mathrm{Im}\left(\delta\colon C^{k-1}_{\mathrm{sing}}(X;G) \to C^{k}_{\mathrm{sing}}(X;G)\right)}$$

These two approaches to cohomology are isomorphic whenever they make sense, for the same reason that they are the same for homology.

Theorem 1.7.12. *The quasi-isomorphism $C_*^{\mathrm{cell}}(X) \to C_*^{\mathrm{sing}}(X)$ induces a quasi-isomorphism $C^*_{\mathrm{sing}}(X;G) \to C^*_{\mathrm{cell}}(X;G)$ and hence an isomorphism*

$$H^*_{\mathrm{sing}}(X;G) \xrightarrow{\cong} H^*_{\mathrm{cell}}(X;G).$$

The advantage of singular cohomology is that it is *contravariant in* X, i.e. for a continuous map $f\colon X \to Y$ we get a map on cohomology in the opposite direction:

$$\begin{aligned}f^*\colon H^k(Y;R) &\longrightarrow H^k(X;R)\\ [\alpha] &\longmapsto f^*[\alpha] = [\alpha\circ f_*]\end{aligned}$$

so $(f^*\alpha)(\sigma) = \alpha(f_*\sigma) = \alpha(f\circ\sigma)$.
Singular cohomology satisfies duals to various theorems that you know for homology. For instance there is excision $H^k(X,A) \cong H^k(X\setminus B, A\setminus B)$, there is a long exact sequence of a pair (though the maps go the other way), a Mayer–Vietoris sequence, etc.

Eilenberg–Mac Lane spaces. In this subsection we do some actual homotopy theory (rather than homological algebra). Namely, we would like to construct a space $K(G,n)$ that "classifies degree n cohomology with coefficients G" in the sense that it satisfies that there is a natural bijection for all CW complexes X

$$H^*(X,A;G) \cong [(X,A),(K(G,n),k_0)]$$

between the (relative) nth cohomology of X with coefficients G and the set of homotopy classes of maps $X \to K(G,n)$.

Observation 1.7.13. If such a mysterious space $K(G,n)$ exists, then it must have the following homotopy groups.

$$\pi_k(K(G,n)) = [(S^k,s_0),(K(G,n),x_0)] \cong H^n(S^k;G) = \begin{cases} G & \text{if } k=n \\ 0 & \text{otherwise.} \end{cases}$$

Let's make this a definition.

Definition 1.7.14. Let $n \geq 1$ and G a group. A space Y is an Eilenberg–Mac Lane space of type (G,n) if its homotopy groups are

$$\pi_k(Y,y_0) \cong \begin{cases} G & \text{if } k=n \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.7.15. If a space X has a contractible universal cover, then it must be an EM-space of type $(G,1)$ where $G = \pi_1(X,x_0)$. Examples for this are:

- S^1 is a $K(\mathbb{Z},1)$,
- $S^1 \times S^1$ is a $K(\mathbb{Z}^2,1)$,
- $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2,1)$.

We will later see that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$.

We can in fact construct $K(G,n)$s. This might be a little surprising, since we don't even know $\pi_3(S^2)$ yet. But it turns out that we can construct these spaces without really having to understand what cells they have.

Theorem 1.7.16. *Let $n \geq 1$ and G a group, and if $n \geq 2$ assume that G is abelian. Then there is a CW complex $K(G,n)$ that is an Eilenberg–Mac Lane space of type (G,n).*

Now we want to show that this EM-space actually has the desired relation to cohomology. First we compute:

Lemma 1.7.17. *There is a canonical isomorphism*

$$H^n(K(G,n);G) \cong \mathrm{Hom}(G,G).$$

We let $\iota \in H^n(K(G,n);G)$ denote the canonical element that corresponds to the identity $\mathrm{id}_G\colon G \to G$. This allows us to define for every X a map

$$\begin{aligned} \kappa\colon [X, K(G,n)] &\longrightarrow H^n(X;G) \\ [f\colon X \to K(G,n)] &\longmapsto f^*\iota. \end{aligned}$$

Proposition 1.7.18. *The map κ is a bijection for all CW complexes X.*

EM-spaces are in fact unique up to weak equivalence.

Corollary 1.7.19. *If Y is any other Eilenberg–Mac Lane space of type (G,n), then there is a weak equivalence $K(G,n) \to Y$.*

Remark 1.7.20. Note that this uniqueness is unusual: normally two spaces with the same homotopy groups don't have to be (weakly) equivalent because there might not be a map between them that induces the isomorphism on homotopy groups. (E.g. $\mathbb{RP}^2 \times S^3$ and $S^2 \times \mathbb{RP}^3$ have the same homotopy groups because they both have $S^2 \times S^3$ as a double-cover, but $H_5(\mathbb{RP}^2 \times S^3) = 0$ whereas $H_5(S^2 \times \mathbb{RP}^3) = \mathbb{Z}$.)

Start lecture 5

1.8 Serre fibrations

Fiber bundles. We have seen that homotopy groups work well with products. To be precise, we have isomorphisms:

$$\pi_k(X \times Y, (x_0, y_0)) \cong \pi_k(X, x_0) \times \pi_k(Y, y_0).$$

The idea of today's lecture is to consider spaces E that "locally look like a product", and to establish a long exact sequence for these.

Definition 1.8.1. Fix a space F. We say that a map $p\colon E \to B$ is a fiber bundle with fiber F if there is an open covering $\{U_i\}_{i\in I}$ of B such that for each i there are homeomorphisms φ_i making the diagram

$$\begin{tikzcd} E \arrow[d, "p"'] & p^{-1}(U_i) \arrow[l, hook'] \arrow[r, "\varphi", "\cong"'] \arrow[d, "p_{|U_i}"'] & U_i \times F \arrow[dl, "\mathrm{pr}_{U_i}"] \\ B & U_i \arrow[l, hook'] & \end{tikzcd}$$

commute. We also say that p is locally trivial.

Example 1.8.2. The Möbius strip is a fiber bundle over S^1 with fiber I. The torus and the Klein bottle are fiber bundles over S^1 with fiber S^1.

Example 1.8.3. There are many examples of fiber bundles:

1. A covering $p\colon E \to B$ is a fiber bundle. (Assuming that B is connected.) Conversely a fiber bundle is a covering if and only if the fiber F is a discrete topological space.

2. The trivial fiber bundle $B \times F \to B$ is a fiber bundle.

3. Assume that M and N are smooth manifolds, N is connected, and $f : M \to N$ is a proper smooth map. Then f is a fiber bundle if and only if it is a submersion (i.e. the differential $D_x(f) : T_xM \to T_{f(x)}N$ is surjective for all $x \in M$.). We will not prove this, but it follows from the tubular neighbourhood theorem in differential geometry.

We will prove the following theorem:

Theorem 1.8.4. *Suppose $p : E \to B$ is fiber bundle with fiber F and $x_0 \in E$ a base-point, write $b_0 := p(x_0)$. Then there is a long exact sequence:*

$$\dots \xrightarrow{\partial_*} \pi_k(F, x_0) \xrightarrow{i_*} \pi_k(E, x_0) \xrightarrow{p_*} \pi_k(B, b_0) \xrightarrow{\partial_*} \pi_{k-1}(F, x_0) \xrightarrow{i_*} \pi_{k-1}(E, x_0) \xrightarrow{p_*} \dots$$

The Hopf fibration. With this in hand let us consider one of the key examples:

Example 1.8.5. Think of S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$. Then there is a map $q : S^{2n+1} \to \mathbb{CP}^n$ defined by sending every unit vector to the complex line it generates. We can also write this as

$$q_n : S^{2n+1} \hookrightarrow \mathbb{C}^{n+1} \setminus \{0\} \twoheadrightarrow (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times = \mathbb{CP}^n.$$

This map is a smooth submersion between closed connected manifolds and hence a fiber bundle. The fiber at a point $[z_0 : \cdots : z_n]$ is the space of unit vectors contained in this complex line, which we may identify with $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

We can also explicitly construct trivialisations as follows. Let $U_i \subset \mathbb{CP}^n$ be the subspace of those $[z_0 : \cdots : z_n]$ with $z_i \neq 0$. These form an open cover of $\mathbb{CP}^n$. Then there is a trivialisation:

$$\begin{aligned} \varphi_i : q_n^{-1}(U_i) &\longrightarrow U_i \times S^1 \\ (z_0, \dots, z_n) &\longmapsto \left([z_0 : \cdots : z_n], \frac{z_i}{|z_i|} \right) \end{aligned}$$

Definition 1.8.6. For $n = 1$ the map constructed above

$$H := q_1 : S^3 \longrightarrow \mathbb{CP}^1 \cong S^2$$

is called the Hopf map.

Exercise 1.8.7. Look up some pictures of the Hopf map!

Remark 1.8.8. The above example can be thought of as an analogue of the quotient map

$$S^n \longrightarrow \mathbb{RP}^n$$

whose fiber is $\{r \in \mathbb{R} \mid |r| = 1\}$. Though in this case the fiber is discrete, so it is just a double-covering. One can also generalise this to the quaternionic case, where one obtains a fiber bundle

$$S^{4n+3} \longrightarrow \mathbb{HP}^n$$

with fiber $\{u \in \mathbb{H} \mid |u| = 1\} \cong \mathrm{SU}_2 \cong S^3$.

Inspecting the long exact sequence we conclude:

Lemma 1.8.9.

$$\pi_k(\mathbb{CP}^n) \cong \begin{cases} 0 & k = 0, 1, \\ \mathbb{Z} & k = 2, \\ \pi_k(S^{2n+1}) & k \geq 3. \end{cases}$$

In the special cases $n = 1$ and $n = \infty$ this means

$$\pi_k(S^2) \cong \begin{cases} 0 & k = 0, 1, \\ \mathbb{Z} & k = 2, \\ \pi_k(S^3) & k \geq 3. \end{cases} \qquad \text{and} \qquad \pi_k(\mathbb{CP}^\infty) \cong \begin{cases} 0 & k \neq 2, \\ \mathbb{Z} & k = 2. \end{cases}$$

Corollary 1.8.10. *The group $\pi_3(S^2) \cong \mathbb{Z}$ is infinite cyclic and generated by the Hopf map*

$$H := q_1 \colon S^3 \hookrightarrow \mathbb{C}^2 \setminus \{0\} \twoheadrightarrow \mathbb{CP}^1 \cong S^2.$$

Corollary 1.8.11. *The space $\mathbb{CP}^\infty$ is an Eilenberg–Mac Lane space of type $(\mathbb{Z}, 2)$.*

To actually establish the existence of the LES, we will have to set up some theory.

The definition of Serre fibrations. First we recall a variant of the homotopy lifting property that you might know from AlgTopI.

Definition 1.8.12. Fix a space X. We say that a map $p\colon E \to B$ has the homotopy lifting property (HLP) with respect to X if any commutative square of solid arrows

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\ X \times [0,1] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

admits a dashed lift as indicated.

Definition 1.8.13. We say that a map $p\colon E \to B$ is a

- Serre fibration if it has the HLP with respect to D^n for all $n \geq 0$.
- Hurewicz fibration if it has the HLP with respect to all spaces X.

Example 1.8.14. There are a few basic examples of fibrations:

1. Every Hurewicz fibration is also a Serre fibration.
2. If $p\colon E \to B$ and $q\colon B \to C$ are Serre fibrations, then so is $q \circ p$.
3. For any two spaces B and F the projection $E = B \times F \to B$ is a Hurewicz (and hence Serre) fibration.
4. If $p\colon E \to B$ is a covering map, then it is a Hurewicz fibration, and in particular a Serre fibration.

5. Later we will show that every fiber bundle is a Serre fibration.

Lemma 1.8.15. *If (X, A) is a CW pair and $p\colon E \to B$ is a Serre fibration, then any commutative square of solid arrows*

$$\begin{tikzcd} X \times \{0\} \cup A \times [0,1] \arrow[r] \arrow[d] & E \arrow[d] \\ X \times [0,1] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

admits a dashed lift as indicated.

When thinking of fibrations, the notion of a pullback is very useful. [This will be mentioned in the next lecture.]

Definition 1.8.16. Given two maps $f\colon C \to B$ and $p\colon E \to B$ we define the pullback of p along f as

$$f^*E \coloneqq C \times_B E = \{(c, e) \in C \times E \mid f(c) = p(e)\}.$$

Alternatively, we can use category theory to define the pullback as the limit of the cospan diagram $f\colon C \to B \leftarrow E : p$:

$$\begin{tikzcd} C \times_B E \arrow[r] \arrow[d, "f^*p \coloneqq \mathrm{pr}_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ C \arrow[r, "f"] & B \end{tikzcd}$$

Lemma 1.8.17. *If $p\colon E \to B$ is a Serre fibration, then for every map $f\colon C \to B$ the pullback of p along f is also a Serre fibration.*

The long exact sequence of a fibration. We now establish the long exact sequence associated to a Serre fibration.

Definition 1.8.18. For a Serre fibration $p\colon E \to B$ and a base-point $b_0 \in B$, we call $F \coloneqq p^{-1}(b_0) \subseteq E$ the fiber of p at b_0.

Theorem 1.8.19. *Suppose $p\colon E \to B$ is a Serre fibration and $x_0 \in E$ a base-point, write $b_0 \coloneqq p(x_0)$. Then the map of pairs $(E, F, x_0) \to (B, \{b_0\}, b_0)$ induces an isomorphism*

$$\pi_k(E, F, x_0) \xrightarrow{\cong} \pi_k(B, b_0)$$

for all $k \geq 0$. As a consequence we have a long exact sequence:

$$\dots \xrightarrow{\partial_*} \pi_k(F, x_0) \xrightarrow{i_*} \pi_k(E, x_0) \xrightarrow{p_*} \pi_k(B, b_0) \xrightarrow{\partial_*} \pi_{k-1}(F, x_0) \xrightarrow{i_*} \pi_{k-1}(E, x_0) \xrightarrow{p_*} \dots$$

Example 1.8.20. [Moved to the exercises.] Suppose $p\colon E \to B$ is a fibration such that the inclusion of the fiber admits a retraction $r\colon E \to F$ with $r \circ i = \mathrm{id}_F$. Then the long exact sequence splits into split short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \pi_k(F) \arrow[r, hook, "i_*"'] & \pi_k(E) \arrow[l, bend right, "r_*"'] \arrow[r, two heads, "p_*"] & \pi_k(B) \arrow[r] & 0 \end{tikzcd}$$

and we can deduce that $\pi_k(E) \cong \pi_k(F) \times \pi_k(B)$. In particular this means that the map

$$(r, p)\colon E \longrightarrow F \times B$$

is a weak homotopy equivalence.
In this case we say that a fibration $p\colon E \to B$ is homotopically trivial.

Locality for Serre fibrations. The criterion of being a Serre fibration can be checked "locally in the base":

Theorem 1.8.21. *Let $p\colon E \to B$ be a map such B admits an open covering $\{U_i\}_{i \in I}$ such that the restrictions*

$$p_{|U_i}\colon p^{-1}(U_i) \longrightarrow U_i$$

are Serre fibrations. Then p is a Serre fibration.

In particular, this shows that fiber bundles are indeed Serre fibrations.

Corollary 1.8.22. *Every fiber bundle is a Serre fibration.*

Start lecture 6

1.9 Mapping spaces and path fibrations

The compact-open topology. A homotopy between two maps $f, g\colon X \to Y$ is defined as a map $H\colon X \times [0,1] \to Y$ restricting to f and g. This definition is useful from a technical perspective, but intuitively it is often simpler to think of H as a *family of maps* $f_t\colon X \to Y$, depending continuously on $t \in [0,1]$ and satisfying $f_0 = f$ and $f_1 = g$. This means we can think of H as a path in the space $\mathrm{Map}(X,Y)$ of maps from X to Y, which starts at f and ends at g. We will define a topology on the mapping space such that there is a correspondence:

$$\{\text{homotopies } H\colon X \times [0,1] \to Y\} \cong \{\text{paths } [0,1] \to \mathrm{Map}(X,Y)\}$$

Definition 1.9.1. For spaces X, Y, we let $\mathrm{Map}(X,Y)$ denote the topological space of continuous maps from X to Y. This space is equipped with the compact-open topology, i.e. the topology generated by the basis

$$W(K,U) = \{f\colon X \to Y \mid f(K) \subseteq U\} \subseteq \mathrm{Map}(X,Y)$$

where $K \subseteq X$ is any compact subset and $U \subseteq Y$ any open subset.

We prove a few basic properties of this topology:

Lemma 1.9.2. *If $f\colon X \times Y \to Z$ is continuous, then its adjunct*

$$\begin{aligned} f^\flat\colon X &\longrightarrow \mathrm{Map}(Y,Z) \\ x &\longmapsto (y \mapsto f(x,y)) \end{aligned}$$

is also continuous. If Y is locally compact, then the converse holds and hence sending f to its adjunct defines a bijection:

$$\alpha\colon \mathrm{Map}(X \times Y, Z) \longrightarrow \mathrm{Map}(X, \mathrm{Map}(Y,Z)).$$

Applying Lemma 1.9.3 to the case $X = I$ we obtain the desired bijection between homotopies and paths in the mapping space. Therefore we have that

$$[X, Y] \cong \pi_0 \mathrm{Map}(X, Y).$$

Lemma 1.9.3. *If moreover Y is also locally compact, then this bijection α is a homeomorphism.* *[Proof omitted.]*

Remark 1.9.4. This lemma is sometimes called the "exponential law for mapping spaces". If we write $Y^X := \mathrm{Map}(X, Y)$, then the exponential law states that there is a homeomorphism

$$Z^{X \times Y} \cong (Z^Y)^X.$$

Mapping spaces also provide us with a good supply of Serre fibrations:

Lemma 1.9.5. *If (X, A) is a CW pair and X is locally compact, then the restriction map*

$$\begin{aligned} \mathrm{res}_A \colon \mathrm{Map}(X, Y) &\longrightarrow \mathrm{Map}(A, Y) \\ f &\longmapsto f_{|A} \end{aligned}$$

is a Serre fibration.

The spaces of maps can be very big, and there is no conceivable CW structure on them. However, the following theorem of Milnor tells us that most mapping space constructions stay in the world of spaces that are homotopy equivalent to a CW complex. We will not prove or use this theorem, but it's there for our peace of mind.

Theorem 1.9.6 (Milnor)**.** *Let (C, C') be a pair of compact spaces and (X, A) a CW pair. Then the space*

$$\mathrm{Map}((C, C'), (X, A)) := \{f \colon C \to X \mid f(C') \subseteq A\} \subseteq \mathrm{Map}(C, X)$$

is homotopy equivalent to a CW complex.

Pointed spaces and the smash product.

Definition 1.9.7. For pointed spaces we define $\mathrm{Map}_*((X, x_0), (Y, y_0)) \subset \mathrm{Map}(X, Y)$ to be the subspace of base-point preserving maps.

We can recover the homotopy groups as:

$$\pi_k(Y, y_0) = \pi_0 \mathrm{Map}_*((S^k, s_0), (Y, y_0)).$$

Definition 1.9.8. The smash product of two pointed spaces (X, x_0) and (Y, y_0) is defined as the quotient

$$X \wedge Y := (X \times Y)/(X \vee Y)$$

where $X \vee Y = X \times \{y_0\} \cup \{x_0\} \cup Y \subset X \times Y$. This is an associative and commutative operation, up to homeomorphism.

Example 1.9.9. The spheres satisfy

$$S^n \wedge S^m \cong S^{n+m}$$

and hence $S^n \cong S^1 \wedge \cdots \wedge S^1$.

Definition 1.9.10. We defined the (reduced) suspension of a based space (X, x_0) as

$$\Sigma X := S^1 \wedge X = S^1 \times X/(S^1 \vee X).$$

Remark 1.9.11. In AlgTopI you saw that suspension shifts the homology groups: $\widetilde{H}_*(\Sigma X) \cong \widetilde{H}_{*-1}(X)$.

With respect to the smash product there is also an exponential law for pointed spaces:

Corollary 1.9.12. *The homeomorphism from Lemma 1.9.3 restricts to a homeomorphism*

$$\mathrm{Map}_*((X, x_0) \wedge (Y, y_0), (Z, z_0)) \cong \mathrm{Map}_*((X, x_0), \mathrm{Map}_*((Y, y_0), (Z, z_0))).$$

Definition 1.9.13. We define the loop space of a based space (X, x_0) as

$$\Omega X := \mathrm{Map}_*((S^1, s_0), (X, x_0)).$$

Applying the exponential law we see that:

$$\mathrm{Map}_*(X, \Omega Y) = \mathrm{Map}_*(X, \mathrm{Map}_*(S^1, Y)) \cong \mathrm{Map}_*(S^1 \wedge X, Y) = \mathrm{Map}_*(\Sigma X, Y).$$

This shows that Σ and Ω form a pair of adjoint functors:

$$\Sigma\colon \mathrm{Top}_* \rightleftarrows \mathrm{Top}_* : \Omega.$$

Remark 1.9.14. Dually to the suspension, the loop space functor moves the homotopy groups one down:

$$\pi_k(\Omega X) \cong \pi_0 \mathrm{Map}_*(S^k, \mathrm{Map}(S^1, X)) \cong \pi_0 \mathrm{Map}_*(S^1 \wedge S^k, X) \cong \pi_{k+1}(X).$$

The path fibration. Motivated by the loop space, we define the path space as:

Definition 1.9.15. The path space of X is defined as

$$PX := \mathrm{Map}([0,1], X).$$

Lemma 1.9.16. *The map* $\mathrm{const}\colon X \to PX$ *that sends a point x to the constant path at x is a homotopy equivalence.*

By Lemma 1.9.5 restricting a path to both endpoints gives a Serre fibration:

$$(\mathrm{ev}_0, \mathrm{ev}_1)\colon PX \longrightarrow X \times X$$

We let $P_{x_0}X$ be the space of paths that start at x_0. Recording their endpoint defines a Serre fibration

$$\mathrm{ev}_1\colon P_{x_0}X \longrightarrow X$$

This is called the path fibration.

Lemma 1.9.17. *The map* const: $X \to PX$ *that sends a point x to the constant path at x is a homotopy equivalence. The space $P_{x_0}X$ is contractible.*

Inspecting the path fibration further we see that its fiber at a point x_1 is the space of paths from x_0 to x_1. In particular, we have a fiber sequence:

$$\Omega X \longrightarrow P_{x_0}X \longrightarrow X$$

The long exact sequence of this fiber sequence gives another proof that $\pi_k\Omega X \cong \pi_{k-1}X$, since $P_{x_0}X$ is contractible.

Start lecture 7

1.10 Cup product and Poincaré duality

So far cohomology looks pretty much the same as homology, but there is additional structure on the cohomology groups that makes them a much stronger invariant than the homology groups.

To define the cup-product we need to assume that our coefficient group G is in fact a commutative ring R.

The Künneth formula. The Künneth formula for cohomology states:

Theorem 1.10.1 (Künneth for cohomology)**.** *Let R be a ring and let X and Y be two spaces such that $H^k(X;R)$ is a finite rank free R-module for all k. Then there is an isomorphism:*

$$\bigoplus_{a+b=n} H^a(X;R) \otimes_R H^b(Y;R) \cong H^n(X \times Y;R)$$

Remark 1.10.2. If $R = \mathbb{Z}, \mathbb{Z}/m, \mathbb{Q}$, then the tensor product over R is the same as the ordinary tensor product of abelian groups/$\mathbb{Z}$-modules. That is, we have $A \otimes B \cong A \otimes_R B$ in these cases. However, for $R = \mathbb{R}, \mathbb{C}$, etc. there is a huge difference between the two. (You do not want to think of $\mathbb{R}$ as an uncountably-dimensional $\mathbb{Q}$ vector space, but $\mathbb{R} \otimes \mathbb{R}$ is huge whereas $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} = \mathbb{R}$.)

Using the isomorphism from the Künneth formula and the diagonal map $\Delta\colon X \to X \times X$ we can define a product on $H^*(X;R)$ as the composite

$$\smile\colon H^k(X;R) \otimes_R H^l(X;R) \longrightarrow H^{k+l}(X \times X;R) \xrightarrow{\Delta^*} H^{k+l}(X;R)$$

The cup product. For clarity, we give a definition from scratch, without using the map from the Künneth formula.

Definition 1.10.3. The cup-product of a singular k-cochain $\alpha \in C^{\mathrm{sing}}(X;R)^k$ and a singular l-cochain $\beta \in C_l^{\mathrm{sing}}(X;R)$ is the singular $(k+l)$-cochain defined as:

$$(\alpha \smile \beta)(\sigma) := \alpha(\sigma_{|\{0,\ldots,k\}}) \cdot \beta(\sigma_{|\{k+1,\ldots,k+l\}}).$$

Here $\sigma_{|S}$ denotes the restriction of $\sigma\colon \Delta^{k+l} \to X$ to the subsimplex spanned by the vertices $S \subset \{0,\ldots,k+l\}$.

The cup product satisfies:

$$\delta(\alpha \smile \beta) = (\delta\alpha) \smile \beta + (-1)^k \alpha \smile (\delta\beta)$$

Using this we can see that the cup product of cocycles is a cocycle, and that the cup product of a cocycle with a coboundary is a coboundary. It follows that the cup product defines a bilinear map (or a linear map if you write $\otimes$ instead of $\times$)

$$\smile : H^k(X;R) \otimes H^l(X;R) \longrightarrow H^{k+l}(X;R)$$

Lemma 1.10.4. *With respect to the cup product the total cohomology*

$$H^*(X;R) := \bigoplus_{k\geq 0} H^k(X;R)$$

is a graded commutative ring. That is, it satisfies $[\alpha] \smile [\beta] = (-1)^{|\alpha|\cdot|\beta|}[\beta] \smile [\alpha]$ *for* $|\alpha|, |\beta| \in \mathbb{N}$ *the degrees of the classes.*

Example 1.10.5. We know that the cohomology $H^k(\mathbb{CP}^n;\mathbb{Z})$ is $\mathbb{Z}$ in even degrees up to n, and 0 otherwise. One can show (and we will do this below) that as a ring this is the truncated polynomial ring

$$H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1}$$

where $\alpha \in H^2(\mathbb{CP}^n;\mathbb{Z})$ is the generator dual to $\mathbb{CP}^1 = S^2$.
A similar result holds for $\mathbb{RP}^n$, if we use $\mathbb{Z}/2$-coefficients.

$$H^*(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\beta]/\beta^{n+1}$$

Here $|\beta| = 1$.

Example 1.10.6. If X and Y are any spaces then we can construct a map

$$\begin{aligned} H^*(X;R) \otimes_R H^*(Y;R) &\longrightarrow H^*(X \times Y;R) \\ \alpha \otimes \beta &\longmapsto (\mathrm{pr}_X^*\alpha) \smile (\mathrm{pr}_Y^*\beta). \end{aligned}$$

This is that map from the Künneth formula, so it is an isomorphism if $H^k(X;R)$ is free of finite rank over R. Moreover, this map is always a ring homomorphism, so it allows us to compute the ring structure on the cohomology of a product of two spaces. For example we have

$$H^*(S^3 \times \mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[\varepsilon,\alpha]/\varepsilon^2$$

where $|\varepsilon| = 3$ and $|\alpha| = 2$.

Example 1.10.7. The product structure of a disjoint union of spaces is not very interesting:

$$H^*(X \amalg Y) \cong H^*(X) \oplus H^*(Y)$$

Accordingly, the cohomology ring of a wedge is the subring where the values on the base-points agree.

$$H^*(X \vee Y) \cong \ker\Big(H^*(X) \oplus H^*(Y) \longrightarrow H^*(\{x_0\}) \oplus H^*(\{y_0\}) \cong R \oplus R \xrightarrow{-} R\Big)$$

For example we have that

$$H^*(S^2 \vee S^4) \cong \mathbb{Z}[\varepsilon_1,\varepsilon_2]/(\varepsilon_1^2,\varepsilon_2^2,\varepsilon_1\varepsilon_2)$$

where $|\varepsilon_1| = 2$ and $|\varepsilon_2| = 4$.
This shows that $\mathbb{CP}^2$ is not homotopy equivalent to $S^2 \vee S^4$. (The two spaces have the same homology and the same cohomology, but their cohomology rings are not isomorphic.)

Relative cup product. Consider a space X with two subspaces $A, B \subset X$ and assume that either A, B are open, or that X is a CW complex with A and B as subcomplexes. Then the cup product can be lifted to a map as follows:

$$\begin{array}{ccc} H^k(X,A;R) \otimes H^l(X,A;R) & \xrightarrow{\smile} & H^{k+l}(X, A \cup B; R) \\ \big\downarrow & & \big\downarrow \\ H^k(X;R) \otimes H^l(X;R) & \xrightarrow{\quad\smile\quad} & H^{k+l}(X;R) \end{array}$$

We can use this to show:

Lemma 1.10.8. *If $\mathbb{CP}^n$ is covered by m contractible open sets $U_1, \ldots, U_m \subset \mathbb{CP}^n$, then $m > n$.*

In fact the lemma only uses that $H^2(U_i) = 0$, as this suffices to lift the generator $c \in H^2(\mathbb{CP}^n)$ to $H^2(\mathbb{CP}^n, U)$.

The Poincaré duality. To construct the Poincaré duality isomorphism, we need the existence of a fundamental class.

Definition 1.10.9. A fundamental class for an n-manifold M is an n-dimensional homology class $[M] \in H_n(M, \partial M)$ such that for each point $p \in M$ the restriction map

$$H_n(M, \partial M) \longrightarrow H_n(M, M \setminus \{p\}) \cong \mathbb{Z}$$

sends it to a generator.

Proposition 1.10.10. *Let M be a compact connected n-manifold. Then there are 2 cases:*

1. *Either M is orientable, admits a fundamental class $[M] \in H_n(M, \partial M)$, and $H_n(M, \partial M) \cong \mathbb{Z}$ is an infinite cyclic group generated by the fundamental class,*
2. *or M is not orientable, does not admit a fundamental class, and $H_n(M, \partial M) = 0$.*

Remark 1.10.11. If we take homology with coefficients in a field $\mathbb{F}_2$ of characteristic 2, then every compact manifold has a fundamental class $[M] \in H_n(M, \partial M; \mathbb{F}_2)$.

Theorem 1.10.12. *Let M be a compact oriented manifold and $\mathbb{F}$ a field. (If $\mathrm{char}(\mathbb{F}) = 2$, the assumption that M is oriented can be dropped.) Then the map*

$$\begin{aligned} H^k(M, \partial M; \mathbb{F}) &\longrightarrow \mathrm{Hom}_{\mathbb{F}}(H^{n-k}(M;\mathbb{F}), \mathbb{F}) \cong H_{n-k}(M;\mathbb{F}) \\ [\alpha] &\longmapsto (\beta \mapsto (\alpha \smile \beta)([M])) \end{aligned}$$

is an isomorphism for all k .

As a consequence we conclude that on a closed oriented manifold M, for every class $\alpha \in H^k(M)$ there must be another class $\beta \in H^{n-k}(M)$ such that $\alpha \smile \beta \neq 0$. In fact, a stronger integral version of this holds:

Corollary 1.10.13 ([Hat02, Corollary 3.40])**.** *Let M be a closed oriented manifold. Then a cohomology class $\alpha \in H^k(M;\mathbb{Z})$ generates an infinite cyclic summand, if and only if there is $\beta \in H^{n-k}(M;\mathbb{Z})$ such that $\alpha \smile \beta \in H^n(M;\mathbb{Z}) \cong \mathbb{Z}$ generates the nth cohomology group. (Equivalently, we can ask that $(\alpha \smile \beta)[M] = \pm 1$.)*

Example 1.10.14. The corollary allows us to prove that $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1}$ as claimed before. See Hatcher, Example 3.40.

1.11 *ΩS^n, Freudenthal suspension, and Hopf invariant 2

[This section will not be covered in the lectures]

The homology of ΩS^n. By studying the path fibration for $X = S^n$ we can prove the following:

Proposition 1.11.1. *The homology of ΩS^n is*

$$H_k(\Omega S^n) \cong \begin{cases} \mathbb{Z} & \text{if } (n-1)|k \\ 0 & \text{otherwise}. \end{cases}$$

The Freudenthal suspension theorem and stable stems. As a consequence we see that the natural map

$$S^{n-1} \to \Omega\Sigma S^{n-1} = \Omega S^n$$

is an isomorphism in homology up to degree $2n-3$, but not surjective in degree $2n-2$. Since the spaces involved are simply connected for $n \geq 3$ (and for $n = 2$ we know that [?]), we conclude from the relative Hurewicz theorem that this map is $(2n-3)$-connected.

This is a special case of the following theorem when $X = S^{n+1}$. (Note that n changes by 2 with respect to the previous example.)

Theorem 1.11.2 (Freudenthal suspension theorem)**.** *Let X be an n-connected space. Then the natural map $X \to \Omega\Sigma X$ is $(2n+1)$-connected. In formulas, the map*

$$\pi_k(X) \to \pi_k(\Omega\Sigma X) \cong \pi_{k+1}(\Sigma X)$$

is an isomorphism for $k \leq 2n$ and a surjection for $k = 2n+1$.

The example has moved to Example 1.13.8.

The Hopf invariant. [See Hatcher 4.B] We can use the cup product to detect the non-triviality of the Hopf map:

Example 1.11.3. In a previous example we show that $\mathbb{CP}^2$ is not homotopy equivalent to $S^2 \vee S^4$. (The two spaces have the same homology and the same cohomology, but their cohomology rings are not isomorphic.) Note that $\mathbb{CP}^2$ is obtained from $\mathbb{CP}^1 = S^2$ by attaching a single 4-cell, so

$$\mathbb{CP}^2 = S^2 \cup_\varphi e^4$$

where $\varphi\colon S^3 \to S^2$ is some attaching map. If two attaching maps are homotopic, then the resulting spaces are homotopy equivalent. (Lemma 1.3.2) Therefore, if φ were homotopic to the constant map const: $S^3 \to S^2$, then $\mathbb{CP}^2$ would be equivalent to $S^2 \cup_{\text{const}} e^4 = S^2 \vee S^4$, which we know to be false. Therefore $\varphi\colon S^3 \to S^2$ must be a non-trivial element of the homotopy group:

$$[\varphi] \neq 0 \in \pi_3(S^2)$$

In fact φ is the Hopf map, and this is another proof that it is non-trivial. A similar argument shows that the $2n$-cell in $\mathbb{CP}^n$ must be attached non-trivially, so we more generally see:

$$[\varphi_n] \neq 0 \in \pi_{2n-1}(\mathbb{CP}^{n-1}).$$

This can be generalised as follows. Given map $f\colon S^{2n-1} \to S^n$ we let $C_f := S^n \cup_f e^{2n}$ denote the CW complex obtained by attaching a $2n$-cell to S^n using f. The cohomology of this space is readily computed as

$$H^0(C(f)) \cong \mathbb{Z}, \qquad H^n(C(f)) \cong \mathbb{Z}\langle\alpha\rangle, \qquad H^{2n}(C(f)) \cong \mathbb{Z}\langle\beta\rangle,$$

where α is the unique class that pulls back to the generator of S^n and β is the class obtained by pulling back the generator under the collapse map $C(f) \to C(f)/S^n \cong S^{2n}$. The ring structure of the cohomology ring is fully determined by $\alpha^2 = ? \cdot \beta$.

Definition 1.11.4. The Hopf invariant of a map $f\colon S^{2n-1} \to S^n$ is the integer $H(f) \in \mathbb{Z}$ such that:

$$\alpha^2 = H(f) \cdot \beta.$$

Example 1.11.5. We argued before that $H(\eta\colon S^3 \to S^2) = 1$ since $C(\eta) = \mathbb{CP}^2$. Similarly, the attaching map of the 8-cell in $\mathbb{HP}^2$ gives rise to a map $f\colon S^7 \to S^4$ of Hopf invariant 1. In fact, one can make also sense of this for the octonions, producing a map $f\colon S^{15} \to S^8$ of Hopf invariant 1.

One can also note that $H(f\colon S^{2n-1} \to S^n) = 0$ whenever n is odd. Since in this case $\alpha^2 = 0$ for degree reasons.

Lemma 1.11.6. *The Hopf invariant is a group homomorphism:*

$$H\colon \pi_{2n-1}(S^n) \longrightarrow \mathbb{Z}$$

Adams proved that maps $f\colon S^{2n-1} \to S^n$ of Hopf invariant 1 only exist if $n = 2, 4, 8$. This has several consequence:

- $\mathbb{R}^n$ is a division algebra only for $n = 1, 2, 4, 8$,
- S^n admits a framing only for $n = 0, 1, 3, 7$,
- The only fiber bundles $S^a \to S^b \to S^c$ that exist are for $(a, b, c) = (0, 1, 1)$, $(1, 3, 2)$, $(3, 7, 4)$, and $(7, 15, 8)$.

Theorem 1.11.7. *For every even $n \geq 2$ there exists a map of Hopf invariant 2. In particular, there is a splitting:*

$$\pi_{2n-1}(S^n) \cong \mathbb{Z} \oplus A$$

for some abelian group A.

To prove this proposition, we will need to compute the cohomology ring of ΩS^{n+1}.

Lemma 1.11.8. *For n odd the cohomology ring of ΩS^n is the free divided power algebra on a generator in degree $(n-1)$. Concretely we have*

$$H^*(\Omega S^n) \cong \mathbb{Z}[\gamma_1, \gamma_2, \ldots]/\left(\gamma_a\gamma_b = \binom{a+b}{a}\gamma_{a+b}\right)$$

where $|\gamma_a| = a \cdot (n-1)$.

Remark 1.11.9. The divided power algebra is also isomorphic to the subring of $\mathbb{Q}[x]$ generated by $\gamma_a = x^a/a!$.

Start lecture 8

1.12 Homotopy fiber sequences and the Puppe sequence

Homotopy fibers. Previously, the mapping cylinder allowed us to think of any map as an inclusion and then the mapping cone is a good replacement for the quotient. Dually, the path fibration allows us to think of any map a fibration. Namely, every map $f\colon X \to Y$ factors as

$$X \xrightarrow{\simeq} X \times_Y PY = \{(x,\gamma) \in X \times PY \mid \gamma(1) = f(x)\} \xrightarrow{\mathrm{ev}_0} Y$$

where the first map is a homotopy equivalence and the second map is a Serre fibration. We now introduce a name for the fiber of this homotopy-replacement.

Definition 1.12.1. We define the homotopy fiber of a map $f\colon X \to Y$ at a point $y_0 \in Y$ as the pullback

$$\mathrm{hofib}_{y_0}(f) \coloneqq X \times_Y P_{y_0}Y = \{(x,\gamma) \in X \times PY \mid \gamma(0) = y_0, \gamma(1) = x\}$$

For any map $f\colon X \to Y$ we hence have a long exact sequence:

$$\ldots \longrightarrow \pi_{k+1}(Y) \xrightarrow{\partial} \pi_k(\mathrm{hofib}_{y_0}(f)) \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{\partial} \ldots$$

Example 1.12.2. Here are a few facts about homotopy fibers.

1. In fact, when $i\colon A \hookrightarrow X$ is a subspace inclusion, then there is a canonical isomorphism:

   $$\pi_k(X, A, x_0) \cong \pi_{k-1}(\mathrm{hofib}_{x_0} i, *).$$

   We could (should???) have defined the relative homotopy groups in this way.

2. The homotopy fiber (at the base-point x_0) of the inclusion of a point $\iota_x\colon \{x\} \hookrightarrow X$ is the space of paths from x_0 to x. In particular for $x = x_0$ we have $\mathrm{hofib}_{x_0}(\iota_x) = \Omega X$.

3. Homotopy fibers are invariant under weak equivalences. More precisely, if we have a commuative square

   $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h_1"'] & B \arrow[d, "h_2"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

   where the h_i are weak equivalences, then for each $b \in B$ the map $k\colon \mathrm{hofib}_b(f) \to \mathrm{hofib}_{h_2(b)}(g)$ is a weak equivalence.

4. If $p\colon E \to B$ is a Serre fibration then the canonical map from the fiber to the homotopy fiber

   $$p^{-1}(b_0) \hookrightarrow \mathrm{hofib}_{b_0}(p)$$

   is a weak homotopy equivalence. (Conversely, we call a map p a quasifibration if this map is a weak equivalence for each $b_0 \in B$. Not every quasifibration is a Serre fibration.)

Lemma 1.12.3. *If $p\colon E \to B$ is a fibration and E is contractible, then the fiber $p^{-1}(b)$ is weakly equivalent to ΩB.*

The Puppe sequence.

Definition 1.12.4. We say that two consecutive maps of pointed spaces

$$F \xrightarrow{i} E \xrightarrow{p} B$$

form a fiber sequence p is a Serre fibration and i restricts to a homeomorphism $F \cong p^{-1}(b_0)$.

Definition 1.12.5. We say that two consecutive maps of pointed spaces

$$F \xrightarrow{i} E \xrightarrow{p} B$$

form a homotopy fiber sequence if i can be factored, up to homotopy, as

$$F \xrightarrow{\simeq} \mathrm{hofib}_{b_0}(p) \longrightarrow E$$

where the first map is a weak equivalence and the second map is $(x, \gamma) \mapsto p(x) = \gamma(1)$.

Lemma 1.12.6. *If $F \to E \to B$ is a homotopy fiber sequence and X is a pointed CW complex, then the sequence:*

$$[X, F]_* \longrightarrow [X, E]_* \longrightarrow [X, B]_*$$

is exact in the middle term.

Lemma 1.12.7. *For any map of pointed spaces $p\colon E \to B$ the homotopy fiber of $\mathrm{hofib}_{b_0}(p) \to E$ is equivalent to the loop space of B.*

Applying this lemma iteratively, we obtain the following diagram, in which the squares commmute up to homotopy.

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{hofib}(k) \arrow[r] & \mathrm{hofib}(j) \arrow[r, "k"] & \mathrm{hofib}(i) \arrow[r, "j"] & \mathrm{hofib}(p) \arrow[r, "i"] & E \arrow[r, "p"] & B \\
\dots \arrow[r] & \Omega F \arrow[r] \arrow[u, "\simeq"] & \Omega E \arrow[r] \arrow[u, "\simeq"] & \Omega B \arrow[r] \arrow[u, "\simeq"] & F \arrow[r] \arrow[u, equal] & E \arrow[r, "p"] \arrow[u, equal] & B \arrow[u, equal]
\end{tikzcd}$$

Theorem 1.12.8 (Puppe sequence)**.** *For every map $p\colon E \to B$ there is a sequence of pointed spaces*

$$\dots \longrightarrow \Omega^2 B \longrightarrow \Omega F \xrightarrow{\Omega i} \Omega E \xrightarrow{\Omega p} \Omega B \longrightarrow F \xrightarrow{i} E \xrightarrow{p} B$$

such that any two maps in this sequence form a homotopy fiber sequence. (Here we write $F := \mathrm{hofib}_{b_0}(p)$.)

Applying the previous lemma to the Puppe sequence we obtain a long exact sequence:

$$\cdots \to [X, \Omega^2 B] \to [X, \Omega F] \to [X, \Omega E] \to [X, \Omega B] \to [X, F] \to [X, E] \to [X, B]$$

In the case $X = S^k$ we obtain another derivation of the LES of a fibration.

***The co-Puppe sequence.** [Not covered in the lectures] Dually to the previous section we can define:

Definition 1.12.9. We say that two consecutive maps of pointed spaces

$$A \xrightarrow{i} B \xrightarrow{q} Q$$

form a homotopy cofiber sequence if q can be factored, up to homotopy, as

$$B \hookrightarrow C(i) \xrightarrow{\simeq} Q$$

where the first map is the inclusion of B into the mapping cone $CA \cup_A B$ and the second map is a homotopy equivalence.

Lemma 1.12.10. *If $A \to B \to C$ is a homotopy cofiber sequence and X is a pointed space, then the sequence:*

$$[C, X]_* \longrightarrow [B, X]_* \longrightarrow [A, X]_*$$

is exact in the middle term.

Theorem 1.12.11 (Barrat-Puppe sequence)**.** *For every map $i\colon A \to B$ there is a sequence of pointed spaces*

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow \Sigma A \xrightarrow{\Sigma i} \Sigma B \xrightarrow{\Sigma j} \Sigma C \longrightarrow \Sigma^2 A \longrightarrow \dots$$

such that any two maps in this sequence form a homotopy cofiber sequence. (Here we write $C := C(i)$.)

As a consequence, we obtain for every X a long exact sequence:

$$\dots \to [\Sigma^2 A, X] \to [\Sigma C, X] \to [\Sigma B, X] \to [\Sigma A, X] \to [C, X] \to [B, X] \to [A, X]$$

Setting $X = K(G, n)$ recovers the long exact sequence for cohomology.

Start lecture 9

1.13 The Blakers–Massey theorem

The Blakers–Massey theorem - Excision for homotopy groups. Recall that the excision theorem for homology may be expressed as follows. Let Y be a space with an covering by two opens $Y = Y_1 \cup Y_2$. Then the map

$$H_k(Y_1, Y_1 \cap Y_2) \longrightarrow H_k(Y, Y_2)$$

induced by the inclusion is an isomorphism for all k. Let us call such a triple (Y, Y_1, Y_2) an open triad and write $Y_0 := Y_1 \cap Y_2$ for the intersection. We can also denote this as a commutative square:

$$\begin{tikzcd} Y_0 \arrow[r, hook, "v"] \arrow[d, hook, "h"'] & Y_1 \arrow[d, hook, "h'"] \\ Y_2 \arrow[r, hook, "v'"] & Y \end{tikzcd}$$

Fun fact: this square is a pullback and a pushout square.

Theorem 1.13.1 (Blakers–Massey). *Suppose (Y, Y_1, Y_2) is an open triad such that $Y_0 \hookrightarrow Y_1$ is p-connected and $Y_0 \hookrightarrow Y_2$ is q-connected for some $p, q \geq 0$. Then the square*

$$\pi_k(Y_1, Y_0) \longrightarrow \pi_k(Y, Y_2)$$

is an isomorphism for $k < p + q$ and surjective for $k = p + q$.

Remark 1.13.2. We will only prove this theorem for $p, q \geq 2$. The more general case is proven in Hatcher, but we will make this additional restriction as it allows us to give nice proof that uses several of the tools we've learnt about so far. (Whereas the proof in Hatcher is a somewhat complicated argument about simplices.)

We can express the theorem in terms of homotopy fibers as follows:

$$\begin{tikzcd} \mathrm{hofib}_{y_0}(v) \arrow[r] \arrow[d, "\tilde{h}"] & Y_0 \arrow[r, hook, "v"] \arrow[d, hook, "h"'] & Y_1 \arrow[d, hook, "h'"] \\ \mathrm{hofib}_{y_0}(v') \arrow[r] & Y_2 \arrow[r, hook, "v'"] & Y \end{tikzcd}$$

The claim of the theorem is equivalent to saying

$$\tilde{h}_*\colon \pi_{k+1}(\mathrm{hofib}_{y_0}(v)) \longrightarrow \pi_{k+1}(\mathrm{hofib}_{y_0}(v'))$$

is an isomorphism for $k + 1 < p + q$ and a surjection for $k + 1 = p + q$. In other words, the theorem says that the map

$$\tilde{h}\colon \mathrm{hofib}_{y_0}(v) \longrightarrow \mathrm{hofib}_{y_0}(v')$$

is $(p + q - 1)$-connected.

Definition 1.13.3. In the above case we say that the square (Y_0, Y_1, Y_2, Y) is $(p + q - 1)$-cartesian.

Reduction to the case that Y is contractible. We will need a few auxiliary lemmas about homotopy fibers.

Lemma 1.13.4. *Let $p\colon E \to B$ be a Serre fibration, $f\colon A \to B$ any map and take the pullback*

$$\begin{tikzcd} A \times_B E \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

Then the canonical map $p'\colon \mathrm{hofib}_{e_0}(\tilde{f}) \to \mathrm{hofib}_{p(e_0)}(f)$ is a weak homotopy equivalence.

In order to reduce to the case where Y is contractible we consider the based path spaces $P \coloneqq P_{y_0}Y$, which is indeed contractible. The evaluation map $\mathrm{ev}_1\colon P \to Y$ is a fibration an we set

$$P_1 \coloneqq \mathrm{ev}_1^{-1}(Y_1) \subseteq P \qquad\qquad P_2 \coloneqq \mathrm{ev}_1^{-1}(Y_2) \subseteq P$$

and hence $P_0 = P_1 \cap P_2 = \mathrm{ev}_1^{-1}(Y_0)$.

Lemma 1.13.5. *It suffices to prove the theorem for the open triad (P, P_1, P_2) instead of (Y, Y_1, Y_2).*

A version for CW pairs. Using the homotopy extension property we can conclude that the Blakers–Massey theorem also applies when Y is a CW complex and $Y_1, Y_2 \subseteq Y$ are subcomplexes.

Theorem 1.13.6 (Blakers-Massey for CW complexes)**.** *Let (X, A) be a p-connected CW pair and (Y, A) a q-connected CW pair. Then the map*

$$\pi_k(X, A) \longrightarrow \pi_k(X \cup_A Y, Y)$$

is an isomorphism for $k < p + q$ and surjective for $k = p + q$.

The Freudenthal suspension theorem. We obtain a more general version of the Freudenthal suspension theorem:

Theorem 1.13.7. *For every n-connected space X the canonical map*

$$X \longrightarrow \Omega\Sigma X$$

is $(2n + 1)$-connected.

Example 1.13.8 (Stable stems)**.** For a fixed k the sequence

$$\pi_k(S^0) \to \pi_{1+k}(S^1) \to \cdots \to \pi_{2k+1}(S^{k+1}) \twoheadrightarrow \pi_{2k+2}(S^{k+2}) \xrightarrow{\cong} \pi_{2k+3}(S^{k+3}) \xrightarrow{\cong} \ldots$$

"stabilies" after $k + 3$ steps. We can therefore define the kth stable stem as the abelian group

$$\pi_k^s := \pi_{k+n}(S^n)$$

where n is any number satisfying $n \geq k + 2$.

Considering the above sequence for $k = 0$ we have

$$\pi_0(S^0) \to \pi_1(S^1) \twoheadrightarrow \pi_2(S^2) \xrightarrow{\cong} \pi_3(S^3) \xrightarrow{\cong} \cdots = \pi_0^s$$

This means that $\pi_n(S^n) \cong \pi_0^s$ is the 0th stable stem and it is a quotient of $\pi_1(S^1) \cong \mathbb{Z}$. We know that it is in fact $\mathbb{Z}$. For $k = 1$ we have

$$\pi_1(S^0) \to \pi_2(S^1) \to \pi_3(S^2) \twoheadrightarrow \pi_4(S^3) \xrightarrow{\cong} \cdots = \pi_1^s$$

We know that $\pi_2(S^1) = 0$ and $\pi_3(S^2) \cong \mathbb{Z}$, so π_1^2 is a quotient of $\mathbb{Z}$. In fact we will later see that it is $\mathbb{Z}/2$. The Freudenthal suspension theorem also implies that the non-trivial element must be the suspension of the Hopf map:

$$[\Sigma^{n-2}\eta\colon S^{n+1} \to S^n] \in \pi_{n+1}(S^n)$$

To calculate the first stable stem we need the Hopf fibration and Steenrod operations.

Start Lecture 10

1.14 Moore–Postnikov towers

Moore–Postnikov factorisations. We now introduce Moore–Postnikov towers, which will allow us to decompose spaces and maps according to their homotopy groups.

Recall that a covering map $X \to Y$ induces an isomorphism of π_k for $k \geq 2$ and an injection on π_1. We can formalise this by saying that covering maps are always 1-truncated:

Definition 1.14.1. A map $f\colon X \to Y$ is n-truncated if, for all $x \in X$, the map

$$f_*\colon \pi_k(X, x) \longrightarrow \pi_k(Y, f(x))$$

is an isomorphism for $k > n$ and an injection for $k = n$.

Proposition 1.14.2. *Let $f\colon (A, a_0) \to (Y, y_0)$ be a map of path-connected spaces. Then there is, for each $n \geq 0$, a factorisation of f as*

$$f\colon A \longrightarrow Z_n \xrightarrow{g_n} Y$$

where the first map is n-connected and the second one is n-truncated. Moreover, we may assume that (Z_n, A) is a relative CW-complex that only has cells of dimension $> n$.

Note that this means we have a factorisation of f as

$$f\colon A \longrightarrow Z_n \xrightarrow{g_n} Y$$

where the first map is an isomorphism homotopy groups of degree $< n$, the second map is an isomorphism in homotopy group of degree $> n$, and in degree n we have the canonical factorisation

$$f_*\colon \pi_n(A, a_0) \twoheadrightarrow \pi_n(Z_n, a_0) \hookrightarrow \pi_n(Y, y_0)$$

of f_* into a surjective map followed by an injective map. (So $\pi_n(Z_n, a_0)$ is in bijection with the image of f_* in $\pi_n(Y, y_0)$.) Also note that g_0 is a weak equivalence.

Definition 1.14.3. We say that the map g is n-truncated.

By applying this to the inclusion of a subspace $i\colon A \hookrightarrow Y$ we obtain the CW-approximation theorem that was claimed in an earlier lecture.

Corollary 1.14.4 (Relative CW approximation)**.** *Let (Y, A) be a pair of spaces. Then there exists a CW pair (X, A) and a map $g\colon X \to Y$ such that g is a weak equivalence and restricts to the identity on A.*

Example 1.14.5. We also obtain another proof of the existence of Eilenberg–Mac Lane spaces. (This is not surprising considering that the proof strategy we used for the Moore–Postnikov factorisation is a generalisation of the one we used for Eilenberg–Mac Lane spaces.) To construct $K(G, n)$, we first let A be an $(n-1)$-connected space with $\pi_n(A) = G$. We can for example take the Moore spaces constructed in the exercises, which additionally satisfy $H_n(A) = G$ and $H_k(A) = 0$ for $k \notin \{0, n\}$, though this extra property won't be needed. Now consider the Moore–Postnikov factorisation of $A \to \mathrm{pt}$. In the $(n+1)$st stage we have that $\pi_k(Z_{n+1}) \cong \pi_k(A) = 0$ for $k < n+1$, and $\pi_k(Z_{n+1}) \to \pi_k(\mathrm{pt})$ is injective for $k \geq n+1$, so it is 0.

Building the tower. Using the compression lemma, we can show that the Moore-Postnikov factorisation is unique in the following sense:

Lemma 1.14.6. *If a map $f\colon A \to Y$ admits two Moore–Postnikov factorisations*

$$A \xrightarrow{i_n} Z_n \xrightarrow{g_n} Y \qquad A \xrightarrow{i'_m} Z'_m \xrightarrow{g'_m} Y$$

with $n \leq m$. Then there is a map $p\colon Z_n \to Z'_m$ making the following diagram commute up to homotopy:

$$\begin{tikzcd}
 & Z_n \arrow[d, "p"] \arrow[dr, bend left, "g_n"] & \\
A \arrow[ur, bend left, "i_n"] \arrow[dr, bend right, "i'_n"'] & Z'_m \arrow[r, phantom] & X \\
 & &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, bend right, "i'_n"'] & Z'_m \arrow[r, bend right, "g'_m"'] & X
\end{tikzcd}$$

If $n = m$, then p is a weak equivalence.

In fact, this follows from the following strengthening of the compression lemma.

Lemma 1.14.7. *Let (Z, A) be a relative CW complex that only has cells of dimension $> n$ and let $g\colon X \to Y$ be an n-truncated map. Then for any commutative diagram:*

$$\begin{tikzcd}
A \arrow[r, "f_0"] \arrow[d, hook] & X \arrow[d, "g"] \\
Z \arrow[r, "f_1"'] \arrow[ur, dashed, "p"] & Y
\end{tikzcd}$$

there exists a dashed map p such that $p_{|A} = f_0$ and such that $g \circ p$ is homotopic to f_1 relative to A.

Therefore we can find maps $p_n\colon Z_n \to Z_{n-1}$ that are the identity on A and satisfying $g_{n-1} \circ p_n \simeq g_n$. By redefining $g_n \coloneqq g_0 \circ p_1 \circ \cdots \circ p_n$ we can even assume that $g_n = g_{n-1} \circ p_n$. Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & Z_3 \arrow[d, "p_3"'] \arrow[ddddr, bend left, "g_3"] & \\
 & Z_2 \arrow[d, "p_2"'] \arrow[dddr, bend left, "g_2"] & \\
 & Z_1 \arrow[d, "p_1"'] \arrow[ddr, bend left, "g_1"] & \\
A \arrow[r, hook] \arrow[ur, hook, bend left] \arrow[uur, hook, bend left] \arrow[uuur, hook, bend left] & Z_0 \arrow[r, "g_0", "\simeq"'] & Y
\end{tikzcd}$$

This is the Moore–Postnikov tower of $f\colon A \to Y$.

The Postnikov tower. An interesting example of the Moore–Postnikov tower is the one obtained from the map $X \to \mathrm{pt}$.

Definition 1.14.8. For $n \geq 0$, the nth Postnikov truncation of X is defined as any $(n+1)$-truncated space $\tau_{\leq n}X$ that receives an $(n+1)$-connected from X:

$$X \longrightarrow \tau_{\leq n}X$$

Remark 1.14.9. We can obtain the nth Postnikov truncation of X as the space Z_{n+1} in the Moore-Postnikov factorisation of the map $X \to \mathrm{pt}$. We could also define $\tau_{\leq -1}X = \mathrm{pt}$ for every space. Note that $\tau_{\leq 0}X = \mathrm{pt}$ if X is connected.

The Postnikov tower of X is the diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \tau_{\leq 3}X \arrow[d] \\
 & \tau_{\leq 2}X \arrow[d] \\
 & \tau_{\leq 1}X \arrow[d] \\
X \arrow[r, hook] \arrow[ur, hook] \arrow[uur, hook] \arrow[uuur, hook] & \tau_{\leq 0}X
\end{tikzcd}$$

The homotopy groups of the nth Postnikov truncation are:

$$\pi_k(\tau_{\leq n}X) \cong \begin{cases} \pi_k(X) & \text{if } k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

Example 1.14.10. We know that the Postnikov tower of the 2-sphere begins as follows:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_4 \arrow[d, "p_3"] \\
 & \mathbb{CP}^\infty \arrow[d] \\
 & \mathrm{pt} \arrow[d, equal] \\
S^2 \arrow[r, hook] \arrow[ur] \arrow[uur] \arrow[uuur, hook] & \mathrm{pt}
\end{tikzcd}$$

Here $X_4 = \tau_{\leq 3}S^2$ is a CW complex with $\pi_2(X_3) \cong \pi_3(X_3) \cong \mathbb{Z}$ and $\pi_k(X_3) = 0$ for $k \notin \{2, 3\}$.
Note that X_4 is not $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$: if it were, then $H_3(X_4) \cong \mathbb{Z}$, but $S^2 \to X_4$ is 3-connected and in particular a surjection on third homology.

Inspecting the long exact sequence, we can compute that the homotopy fibers of the maps p_n must be Eilenberg–Mac Lane spaces. To be precise, we have:

$$\operatorname{hofib}\left(:\ \tau_{\leq n}X \to \tau_{\leq n-1}X\right) \simeq K(\pi_n X, n)$$

The Whitehead tower. [I changed the indexing in this section a bit. I hope it makes more sense now.] Dually, we also obtain an interesting tower from the base-point inclusion $\mathrm{pt} \to X$ of a pointed space.

Definition 1.14.11. For $n \geq 0$, the n-connected cover of a pointed space X is defined as any n-connected space $\tau_{>n}X$ equipped with an n-truncated map to X:

$$\tau_{>n}X \longrightarrow X.$$

Remark 1.14.12. We can obtain the n-connected cover of X as the space Z_n in the Moore-Postnikov factorisation of the base-point inclusion $\mathrm{pt} \longrightarrow X$. If we assume that X is path connected, then $\tau_{>0}X = X$. (If X is not connected it would make sense to define $\tau_{\geq 1}X$ as the path component of the base-point.) We could define $\tau_{>-1} := X$. (The identity map id_X is (-1)-truncated, i.e. a weak equivalence.)

This is the Whitehead tower of X:

$$\begin{tikzcd}
\vdots \arrow[d] & \\
\tau_{>3}X \arrow[d] \arrow[ddddr, bend left, "g_3"] & \\
\tau_{>2}X \arrow[d] \arrow[dddr, bend left, "g_2"] & \\
\tau_{>1}X \arrow[d] \arrow[ddr, bend left, "g_1"] & \\
\tau_{>0}X \arrow[r, "g_0"] & X
\end{tikzcd}$$

The homotopy groups of the nth Whitehead cover are:

$$\pi_k(\tau_{>n}X) \cong \begin{cases} \pi_k(X) & \text{if } k > n \\ 0 & \text{if } k \leq n. \end{cases}$$

Remark 1.14.13. Even though we call X_n a "cover" X, the map $g_n \colon X_n \to X$ is not a covering map. However, for $n = 1$, the 1-connected cover $\tau_{>1}X \to X$ is weakly equivalent to the universal cover of X. Hence the name.

Again we can inspect the long exact sequence, and observe that the homotopy fibers of the maps in the tower must be Eilenberg–Mac Lane spaces. To be precise, we have:

$$\operatorname{hofib}\left(\tau_{>n}X \to \tau_{>n-1}X\right) \simeq K(\pi_n X, n-1)$$

For example the map $\tau_{>1}X \to \tau_{>0}X \simeq X$ has as fibers $K(\pi_1 X, 0)$, i.e. the discrete set $\pi_1 X$. This makes sense since the 1-connected cover is equivalent to the universal cover.

Remark 1.14.14. Every connected space X fits into homotopy fiber sequences:

$$\tau_{>n} X \longrightarrow X \longrightarrow \tau_{\leq n} X$$

for all n.

Example 1.14.15. We know that the Whitehead tower of the 2-sphere begins as follows:

$$\begin{tikzcd}
\vdots \arrow[d] & \\
X_3 \arrow[d, "p_3"'] \arrow[dddr, bend left, "g_3"] & \\
S^3 \arrow[d, "\eta"'] \arrow[ddr, bend left, "\eta"] & \\
S^2 \arrow[d, equal] \arrow[dr, bend left, "\mathrm{id}"] & \\
S^2 \arrow[r, equal] & S^2
\end{tikzcd}$$

Here $X_3 = \tau_{>3} S^2 \simeq \tau_{>3} S^3$ is the 3-connected cover of the 2-sphere, or equivalently of the 3-sphere. One way of constructing it is to take the map $f \colon S^3 \to K(\mathbb{Z}, 3)$ that picks the generator of $\pi_3 K(\mathbb{Z}, 3) = \mathbb{Z}$, and then set $X_3 = \mathrm{hofib}(f)$.

Generalising the last observation of the above example we get:

Lemma 1.14.16. *If $\{\tau_{>n} X\}$ is the Whitehead tower of X, then for each n there is a map f_n such that*

$$\tau_{>n} X \simeq \mathrm{hofib}\left(f_n \colon \tau_{>n-1} X \to K(\pi_n X, n)\right)$$

Proof. We would like to construct a (based) map

$$f_n \colon \tau_{>n-1} X \longrightarrow K(\pi_n X, n).$$

Because Eilenberg–Mac Lane spaces represent cohomology, homotopy classes of such maps correspond to elements in:

$$[\tau_{>n-1} X, K(G, n)] \cong H^n(\tau_{>n-1} X; G)$$

where $G = \pi_n X$. There is a map

$$\mathrm{Hom}(H_n(\tau_{>n-1} X), G) \longrightarrow H^n(\tau_{>n-1}; G)$$

and $H_n(\tau_{>n-1} X) \cong \pi_n X = G$ by Hurewicz. So the identity map id_G defines a cohomology element

$$[\mathrm{id}_G] \in H^n(\tau_{>n-1} X; G)$$

which corresponds to the desired map $f_n \colon \tau_{>n-1} X \longrightarrow K(\pi_n X, n)$. By construction f_n induces the identity on homotopy groups, so we can use the LES to identify the homotopy fiber $\mathrm{hofib}(f_n)$ with $\tau_{>n-1} X$. $\square$

***Principal fibrations in the Postnikov tower.** In the previous section we showed that the maps in the Whitehead tower can be written as the homotopy fiber of maps $\tau_{>n-1}X \to K(\pi_n X, n)$. This can be quite a useful property, particularly if we were to talk about obstruction theory... In general we can make the following definition.

Definition 1.14.17 ([Hat02, p. 412])**.** We say that a map $p\colon E \to B$ is a principal fibration if there is a homotopy fiber sequence

$$E \xrightarrow{p} B \xrightarrow{f} X$$

for some space X.

Note that in particular this means that $\mathrm{hofib}(p) \simeq \Omega X$ by the Puppe sequence. This is already a strong obstruction to a generic map being a principal fibration, as most spaces are not the loop space of any space. (For instance if $\pi_1\mathrm{hofib}(p)$ is not abelian we cannot possibly have p be principal, as otherwise $\pi_1\mathrm{hofib}(p) \cong \pi_2(X)$.)

With this in mind Lemma 1.14.16 may be stated as saying that the maps $\tau_{\geq n+1}X \to \tau_{\geq n}X$ in the Whitehead tower are *always* principal fibrations. In the case of the Postnikov tower the situation is a little more complicated.

Proposition 1.14.18 ([Hat02, Theorem 4.69])**.** *The map $\tau_{\leq n}X \longrightarrow \tau_{\leq n-1}X$ in the Postnikov tower is a principal fibration if and only if the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial.*

1.15 *Obstruction theory

1.16 *Steenrod operations

Start lecture 11

2 Spectral Sequences

2.1 Using the Leray–Serre spectral sequence

The Leray–Serre spectral sequence is a tool for computing the homology of the total space of a (homotopy) fiber sequence from the homology of the base and the fiber. Given a (homotopy) fiber sequence

$$F \longrightarrow E \longrightarrow B$$

where we assume that $\pi_1(B)$ acts trivially on $H_*(F)$, the Leray–Serre spectral sequence has signature

$$E^2_{p,q} = H_p(B; H_q(F;G)) \Rightarrow H_{p+q}(E;G)$$

where G is any abelian group. In words we say that the E^2-page of the spectral sequence is $H_p(B; H_q(F;G))$ (the homology of the base with coefficients in the homology of the fiber) and the spectral sequence converges to the homology of the total space.

Definition. We now introduce the algebraic definition of a spectral sequence.

Definition 2.1.1. A spectral sequence $(E_{\bullet,\bullet}, d)$ (in homological Serre grading), starting on page $r_0 \geq 1$, consists of:

1. for each $r \geq r_0$ a bigraded abelian group $(E^r_{p,q})_{p,q\in\mathbb{Z}}$ called the rth page of the spectral sequence.
2. For all $r \geq r_0$ and $p, q \in \mathbb{Z}$ a map of abelian groups
$$d^r_{p,q} \colon E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$$
called the rth differential or d^r-differential, which squares to zero in the sense that
$$d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$$
holds for all p, q, r.
3. For all $r \geq r_0$ and $p, q \in \mathbb{Z}$, isomorphisms of abelian groups
$$E^{r+1}_{p,q} \cong \frac{\ker\left(d^r_{p,q} \colon E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}\right)}{\operatorname{im}\left(d^r_{p+r,q-r+1} \colon E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q}\right)}$$
For simplicity these two groups are usually assumed to be equal.

Example 2.1.2. As an example of what a spectral sequences looks like we consider the case of the Hopf fibration:
$$S^1 \longrightarrow S^3 \longrightarrow S^2$$
Here the spectral sequence has signature
$$E^2_{p,q} = H_p(S^2, H_q(S^1)) \Rightarrow H_{p+q}(S^3).$$
This means that the second page of this spectral sequence looks as follows:

$$\begin{array}{c|ccc} 1 & \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 & \mathbb{Z} \\ \hline & 0 & 1 & 2 \end{array}$$

The arrow indicates a non-trivial d_2-differential. To obtain the next page of the spectral sequence we take the homology with respect to this differential. This "kills" the copies of $\mathbb{Z}$ at the source and target of the differential, resulting in the following E^3-page:

$$\begin{array}{c|ccc} 1 & 0 & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 & 0 \\ \hline & 0 & 1 & 2 \end{array}$$

From this we can read of the homology of S^3.

Remark 2.1.3. When passing to the next page the groups always "get smaller". To be precise, $E^{r+1}_{p,q}$ is always a sub-quotient of $E^r_{p,q}$. In the finitely generated case this for example means that the rank can only get smaller.

The Serre spectral sequence has the following convenient feature:

Definition 2.1.4. We say that $(E_{\bullet,\bullet}, d)$ is a first quadrant spectral sequence if

$$E^{r_0}_{p,q} = 0$$

whenever $p < 0$ or $q < 0$.

In a first quadrant spectral sequence every entry $E^r_{p,q}$ stabilises for large r. Concretely we have

$$E^{r+1}_{p,q} \cong E^r_{p,q}$$

whenever $r > \max(p, q+1)$, because $d^r_{p,q}$ maps to the 0-group (so its kernel is all of $E^r_{p,q}$) and $d^r_{p+r,q+r-1}$ comes from the 0-group (so its image is 0). Therefore the following definition makes sense:

Definition 2.1.5. For a first quadrant spectral sequence we define the E^∞-page as:[1]

$$E^\infty_{p,q} := E^r_{p,q} \quad \text{for } r \gg p, q.$$

Computations over a field. The declared purpose of the Serre spectral sequence is to compute the homology of the total space of a fibration. The simplest situation is when the E^∞ page consists of finite free $\mathbb{Z}$-modules, in this case there is an isomorphism

$$H_n(E) \cong \bigoplus_{p+q=n} E^\infty_{p,q}$$

Theorem 2.1.6. *For every field $\mathbb{F}$ and every fiber sequence*

$$F \longrightarrow E \longrightarrow B$$

such that $\pi_1(B)$ acts trivially on $H_*(F;\mathbb{Q})$ there is a (natural) Leray–Serre spectral sequence of signature

$$E^2_{p,q} = H_p(B; H_q(F;\mathbb{F})) \Rightarrow H_{p+q}(E;\mathbb{F})$$

meaning that the $E^2_{p,q}$ is given by

$$E^2_{p,q} = H_p(B; H_q(F;\mathbb{F})) \cong H_p(B;\mathbb{F}) \otimes_{\mathbb{F}} H_q(F;\mathbb{F})$$

and the homology of E can be computed as:[2]

$$H_n(E;\mathbb{F}) \cong \bigoplus_{p+q=n} E^\infty_{p,q}.$$

[1]To be precise: once there cannot be any differentials out of $E^r_{p,q}$ there are canonical quotient maps $E^r_{p,q} \to E^{r+1}_{p,q}$ and we define $E^\infty_{p,q}$ to be the colimit over these. However this colimit stabilises after $r > p+q+1$, so we don't need to worry about it.

[2]Warning: this isomorphism is not natural.

This already allows us to make some simple computations:

Example 2.1.7. Suppose we have a fiber sequence

$$S^2 \longrightarrow X \longrightarrow S^2.$$

After computing the E^2-page we see that there is no room for d^2 differentials.

$$\begin{array}{c|ccc}
2 & \mathbb{Q} & 0 & \mathbb{Q} \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Q} & 0 & \mathbb{Q} \\
\hline
 & 0 & 1 & 2
\end{array}$$

So the E^3 page looks the same as the E^2 and again there's no room for differentials. Continuing indefinitely, we compute the E^∞-page and read off:

$$H_k(X;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & k = 0, 4 \\ \mathbb{Q}^2 & k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

This means that X has the same rational homology as $S^2 \times S^2$, though we can't be sure that it is actually equivalent to $S^2 \times S^2$.

Example 2.1.8. Generalising the above example we see that for any fiber sequence

$$S^n \longrightarrow X \longrightarrow S^m$$

with $n \neq m - 1$ there is an isomorphism

$$H_*(X;\mathbb{Q}) \cong H_*(S^n \times S^m;\mathbb{Q}).$$

Example 2.1.9. On the other hand, in the case of the quaternionic Hopf fibration

$$S^3 \to S^7 \to S^4 = \mathbb{HP}^1$$

we can see that $E^2 = E^3 = E^4$ as there's no room for d^2 or d^3 differentials, but there must be a d^4 differential.

$$\begin{tikzcd}
3 & \mathbb{Q} & & & & \mathbb{Q} \\
2 & & & & & \\
1 & & & & & \\
0 & \mathbb{Q} & & & & \mathbb{Q} \arrow[uuullll] \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}
\qquad
\begin{array}{c|ccccc}
3 & & & & & \mathbb{Q} \\
2 & & & & & \\
1 & & & & & \\
0 & \mathbb{Q} & & & & \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

Then from the E^5 page on there's no more room for differentials and so we end up with the homology of S^7 as expected.

Convergence for integral coefficients. We would like to also use the spectral sequence for integral coefficients, but this will mean that we have to be more careful about what we mean by "convergence" of the spectral sequence. Often we'll have to deal with spectral sequences where there are torsion groups on the E^∞-page.

Example 2.1.10 ([Tan20, Step 8])**.** Consider the following fiber sequence

$$S^1 \longrightarrow \mathbb{RP}^3 \xrightarrow{\overline{\eta}} \mathbb{CP}^1$$

where $\overline{\eta}$ is obtained from the Hopf map $\eta\colon S^3 \to S^2$ by observing that it sends antipodal points to the same value and hence descends to $\mathbb{RP}^3$. The second page of this spectral sequence looks as follows:

$$\begin{tikzcd}
1 & \mathbb{Z} & & \mathbb{Z} \\
0 & \mathbb{Z} & & \mathbb{Z} \arrow[ull, "\cdot 2"'] \\
 & 0 & 1 & 2
\end{tikzcd}$$

Now there must be a d_2 differential, and it must be given by multiplication by 2 (or by -2), since we know that $H_1(\mathbb{RP}^3) = \mathbb{Z}/2$. This results in the following E^3-page:

$$\begin{array}{c|ccc}
1 & \mathbb{Z}/2 & & \mathbb{Z} \\
0 & \mathbb{Z} & & 0 \\
\hline
 & 0 & 1 & 2
\end{array}$$

As usual we can read off the homology of the total space by looking at anti-diagonals. This indicates that this also works in the presence of torsion.

The precise statement for how to read of the homology of the total space off the E^∞ page is as follows.

Theorem 2.1.11. *For every abelian group G and every fiber sequence*

$$F \longrightarrow E \longrightarrow B$$

such that $\pi_1(B)$ acts trivially on $H_*(F;G)$ there is a (natural, convergent) Leray–Serre spectral sequence of signature

$$E^2_{p,q} = H_p(B; H_q(F;G)) \Rightarrow H_{p+q}(E;G)$$

meaning that the $E^2_{p,q}$ *is given by* $E^2_{p,q} = H_p(B; H_q(F;G))$ *and there is a natural filtration*

$$0 = F^n_{-1} \subseteq F^0_n \subseteq \cdots \subseteq F^n_n = H_n(E;\mathbb{Z})$$

and natural short exact sequences:

$$0 \to F^{p+q}_{p-1} \hookrightarrow F^{p+q}_p \twoheadrightarrow E^\infty_{p,q} \to 0$$

Remark 2.1.12. In general we can't expect $H_p(B; H_q(F;\mathbb{Z}))$ to be $H_p(B;\mathbb{Z}) \otimes H_q(F;\mathbb{Z})$ because there is a Tor-term in the Künneth formula. However, if we assume that either $H_p(B)$ or $H_q(F)$ is a finite free abelian group (i.e. is equivalent to some $\mathbb{Z}^r$), then there is no Tor-term and we can assume:

$$E^2_{p,q} = H_p(B) \otimes H_q(F).$$

In particular this means the following:

Corollary 2.1.13. *If* $H_n(E) = 0$*, then* $E^\infty_{p,n-p} = 0$ *holds for all* p*.*

Corollary 2.1.14. *If for each* n *only one of the groups*

$$E^\infty_{0,n}, E^\infty_{1,n-1}, \ldots, E^\infty_{n,0}$$

is non-zero, then $H_n(X)$ *is isomorphic to this group. If only two of them are non-zero, say* $E^\infty_{p,n-p}$ *and* $E^\infty_{p',n-p'}$ *with* $p < p'$*, then there is a short exact sequence:*

$$0 \longrightarrow E^\infty_{p,n-p} \longrightarrow H_n(X) \longrightarrow E_{p',n-p'} \longrightarrow 0$$

We can explore what this means in the following example:

Example 2.1.15 ([Tan20, Step 9])**.** Consider the following fiber sequence where U_2 is the group of unitary 2×2-matrices.[3]

$$S^1 \longrightarrow U_2 \longrightarrow \mathbb{RP}^3$$

Hwe will use as a black-box that $\pi_1(\mathbb{RP}^3)$ acts trivially on $H_1(S^1)$. Its E^2-page is of the form:

$$\begin{tikzcd}
1 & \mathbb{Z} & \mathbb{Z}/2 & & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z}/2 & & \mathbb{Z} \arrow[ull, "?"'] \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

There is potential for one non-trivial differential. (And it turns out that this differential is in fact non-trivial.) Independently of this differential, we can try to read off the first homology of U_2. The above corollary tells us that there is a short exact sequence:

$$0 \to \mathbb{Z} \longrightarrow H_1(U_2) \longrightarrow \mathbb{Z}/2 \to 0$$

[3]To construct this fiber sequence identify $S^1 = U_1$ and consider the inclusion $U_1 \hookrightarrow U_2$. Then observe that $U_2/U_1 \cong \mathbb{RP}^3$.

So there are two options for what $H_1(U_2)$ might be: either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$. It turns out (by using the homeomorphism $\mathrm{U}_2 \cong \mathrm{U}_1 \times \mathrm{SU}_2$) that $\pi_1(U_2) = \mathbb{Z}$, so we must be in the first case here. This means that the filtration of $H_1(U_2)$ is

$$0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} = H_1(U_2)$$

with associated graded $E^\infty_{1,0} = \mathbb{Z}/2$ and $E^\infty_{0,1} = \mathbb{Z}$.

Start lecture 12

Arguing backwards. Many arguments can be made by using the Serre spectral sequence in reverse by using Corollary 2.1.13.

Example 2.1.16. Consider the Serre spectral sequence for the fiber sequence

$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n.$$

The E_2-page looks as follows. (Here I drew $n = 3$.)

$$\begin{array}{c|ccccccc}
1 & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\
0 & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

There must be non-trivial d^2 differentials in all three spots where it's possible, because we know that the spectral sequence converges to the homology of S^{2n+1}.

Suppose for a moment that we didn't know the homology of $\mathbb{CP}^n$, but we knew it's simply connected. Then the spectral sequence looks like this:

$$\begin{array}{c|ccccccc}
1 & \mathbb{Z} & & H^2(\mathbb{CP}^3) & H^3(\mathbb{CP}^3) & H^4(\mathbb{CP}^3) & H^5(\mathbb{CP}^3) & H^6(\mathbb{CP}^3) \\
0 & \mathbb{Z} & & H^2(\mathbb{CP}^3) & H^3(\mathbb{CP}^3) & H^4(\mathbb{CP}^3) & H^5(\mathbb{CP}^3) & H^6(\mathbb{CP}^3) \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

By arguing inductively we can see that the only way this spectral sequence can converge to the homology of S^{2n+1} is if $H^{2i}(\mathbb{CP}^n) = \mathbb{Z}$ and $H^{2i+1}(\mathbb{CP}^n) = \mathbb{Z}$.

Using the similar argument on the fiber sequence

$$\Omega S^n \longrightarrow P_{s_0} S^n \longrightarrow S^n$$

can compute the homology of the loop space ΩS^n.

Lemma 2.1.17. *The homology of ΩS^n is given by:*

$$H_k(\Omega S^n) \cong \begin{cases} \mathbb{Z} & \text{if } (n-1) \text{ divides } k \\ 0 & \text{otherwise.} \end{cases}$$

Naturality of spectral sequences. So far we have essentially now tools for determining differentials, other than deducing them from what the spectral sequence should converge to. We can however sometimes figure out differentials in one spectral sequence by comparing it to another.

Example 2.1.18. Recall the mod 2 Hopf fiber sequence from Example 2.1.10. There is a map of fiber sequences:

$$\begin{tikzcd}
S^1 \arrow[r, hook] \arrow[d, "\cdot 2"'] & S^3 \arrow[r, "\eta"] \arrow[d, two heads] & S^2 \arrow[d, equal] \\
S^1 \arrow[r, hook] & \mathbb{RP}^3 \arrow[r, "\overline{\eta}"] & S^2
\end{tikzcd}$$

and this induces a map of spectral sequences. Let us write $(D_{\bullet,\bullet}, d)$ for the spectral sequence coming from the Hopf fibration and $(E_{\bullet,\bullet}, d)$ for the spectral sequence coming from the mod 2 Hopf fibration. Then on E^2-pages the map $D^2_{\bullet,\bullet} \to E^2_{\bullet,\bullet}$ is

$$\begin{tikzcd}[row sep=small]
1 & \mathbb{Z} & 0 & \mathbb{Z} \\
0 & \mathbb{Z} & 0 & \mathbb{Z} \arrow[ull] \\
 & 0 & 1 & 2
\end{tikzcd}
\longrightarrow
\begin{tikzcd}[row sep=small]
1 & \mathbb{Z} & & \mathbb{Z} \\
0 & \mathbb{Z} & & \mathbb{Z} \arrow[ull, "\cdot 2"'] \\
 & 0 & 1 & 2
\end{tikzcd}$$

This being a map of spectral sequences means that the map on the E^2-page commutes with the d^2-differential. In particular we have a commuative square:

$$\begin{tikzcd}
D^2_{2,0} \arrow[r, "\cong"] \arrow[d, "d^2_D"'] & E^2_{2,0} \arrow[d, "d^2_E"] \\
D^2_{0,1} \arrow[r, "\cdot 2"] & E^2_{0,1}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
H_2(S^2) \arrow[r, "\mathrm{id}"] \arrow[d, "d^2_D"'] & H_2(S^2) \arrow[d, "d^2_E = \mathrm{id}"] \\
H_1(S^1) \arrow[r, "\cdot 2"] & H_1(S^1)
\end{tikzcd}$$

Since we know that the map of fiber sequence is the degree 2 map on the fiber and the identity on the base, we can fill in the vertical maps. Suppose moreover that we already know that the differential in the Hopf fibration is an isomorphism (since S^3 is simply connected), then we can conclude from this square that the differential in $E^2_{\bullet,\bullet}$ must be multiplication by 2.

A more interesting example of this technique is the following:

Example 2.1.19. The free loop space of $\mathbb{CP}^\infty$ is defined as

$$L\mathbb{CP}^\infty := \mathrm{Map}(S^1, \mathbb{CP}^\infty).$$

Note that unlike in the case of the loop space we do not require the map $S^1 \to \mathbb{CP}^\infty$ to preserve the basepoint. There is a fiber sequence:

$$\Omega\mathbb{CP}^\infty \longrightarrow L\mathbb{CP}^\infty \longrightarrow \mathbb{CP}^\infty$$

where the fibration is given by evaluating a loop at the basepoint. The E^2-page of the spectral sequence looks as follows (for $n = 3$)

$$\begin{tikzcd}
1 & \mathbb{Z} & ? & \mathbb{Z} & ? & \mathbb{Z} & ? & \mathbb{Z} \\
0 & \mathbb{Z} & & \mathbb{Z} \arrow[llu] & & \mathbb{Z} \arrow[llu] & & \mathbb{Z} \arrow[llu] \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}$$

This is the same as the for the spectral sequence associated to $\Omega\mathbb{CP}^\infty \to * \to \mathbb{CP}^\infty$, but we would expect the differentials to be different this time. In fact, on can show that all the differentials must be trivial. To see this consider the following map of fiber sequences:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathbb{CP}^\infty \arrow[r, equal] \arrow[d, "\mathrm{const}"'] & \mathbb{CP}^\infty \arrow[d, equal] \\
\Omega\mathbb{CP}^\infty \arrow[r, hook] & L\mathbb{CP}^\infty \arrow[r, "\mathrm{ev}"] & \mathbb{CP}^\infty
\end{tikzcd}$$

Here $\mathrm{const}\colon \mathbb{CP}^\infty \to L\mathbb{CP}^\infty$ is the map that sends a point to the constant loop at this point. The top fiber sequence is trivial and in particular all differentials must be trivial. The map of fiber sequences induces a map of spectral sequences $(D_{\bullet,\bullet}, d) \to (E_{\bullet,\bullet}, d)$:

$$\begin{tikzcd}
0 & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}
\longrightarrow
\begin{tikzcd}
1 & \mathbb{Z} & ? & \mathbb{Z} & ? & \mathbb{Z} & ? & \mathbb{Z} \\
0 & \mathbb{Z} & & \mathbb{Z} \arrow[llu] & & \mathbb{Z} \arrow[llu] & & \mathbb{Z} \arrow[llu] \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}$$

This being a map of spectral sequences means that it commutes with the differentials. So in particular we have commutative squares:

$$\begin{tikzcd}
D^2_{2i,0} \arrow[r, "\cong"] \arrow[d, "d^2"'] & E^2_{2i,0} \arrow[d, "d^2"] \\
D^2_{2i-2,1} \arrow[r] & E^2_{2i-2,1}
\end{tikzcd}$$

But $D^2_{2i-2,1} = 0$ and the map $D^2_{\bullet,0} \to E^2_{\bullet,0}$ is an isomorphism, so it follows that all d^2-differentials in $(E_{\bullet,\bullet}, d)$ are trivial. Therefore the spectral sequence collapses and we get

$$H_k(L\mathbb{CP}^\infty) \cong \mathbb{Z}$$

for all $k \geq 0$. (In fact one can show that $L\mathbb{CP}^\infty \simeq S^1 \times \mathbb{CP}^\infty$, giving a more satisfying explanation for this computation.)

The 0**th row and** 0**th column.** The previous example also illustrates the special role played by the 0th row of the spectral sequence $E^2_{p,0} = H_p(B)$. Note that since no differential could ever hit $E^r_{p,0}$, when passing to the next page, all we need to do is to take the kernel of d^r. This means that there is a decreasing filtration:

$$E^2_{p,0} \supseteq E^3_{p,0} \supseteq \cdots \supseteq E^{p+1}_{p,0} = E^{p+2}_{p,0} = \cdots = E^\infty_{p,0}.$$

Moreover, we know from the theorem about convergence that there is a surjective map $H_p(E) = F^p_p \twoheadrightarrow E^\infty_{p,0}$. In summary we have a map

$$H_p(E) \twoheadrightarrow E^\infty_{p,0} \hookrightarrow E^2_{p,0} = H_p(B).$$

Lemma 2.1.20. *This is exactly the map induced on homology by the fibration $E \to B$. Therefore, the image of $H_p(E) \to H_p(B)$ is exactly the intersection of all the kernels of differentials leaving $E^r_{p,0}$. In particular, $H_p(E) \to H_p(B)$ is surjective if and only if all differentials leaving $E^r_{p,0}$ are trivial.*

Example 2.1.21. Consider the map $f\colon K(\mathbb{Z},2) \to K(\mathbb{Z},2)$ that induces multiplication by 2 on π_2. We will show that $f_*\colon H_{2k}(K(\mathbb{Z},2)) \to H_{2k}(K(\mathbb{Z},2))$ is multiplication by 2^k, without using the ring structure on cohomology. There is a homotopy fiber sequence

$$K(\mathbb{Z}/2,1) \longrightarrow K(\mathbb{Z},2) \xrightarrow{f} K(\mathbb{Z},2).$$

By studying the Serre spectral sequence of this we see that $E^\infty_{2k,0} \subset E^2_{2k,0} = \mathbb{Z}$ is the subgroup $2^k\mathbb{Z} \subset \mathbb{Z}$. This implies that the image of f_* on H_{2k} is $2^k\mathbb{Z}$, which is what we wanted to show.

[The 0th column wasn't discussed in the lectures, but is analogous to the 0th row.] Dually, we can study the 0th column $E^2_{0,q} = H_q(F)$. Here all differentials leaving the group must be trivial and hence when passing to the next page we are always taking a quotient. This means that there is a sequence of quotient maps.

$$E^2_{0,q} \twoheadrightarrow E^3_{0,q} \twoheadrightarrow \cdots \twoheadrightarrow E^{q+2}_{0,q} = E^{q+3}_{0,q} = \cdots = E^\infty_{0,q}.$$

Moreover, we know from the theorem about convergence that there is an injective map $E^\infty_{0,q} = F^q_0 \hookrightarrow F^q_q = H_q(E)$. In summary we have a map

$$H_q(F) = E^2_{0,q} \twoheadrightarrow E^\infty_{0,q} \hookrightarrow H_q(E).$$

Lemma 2.1.22. *This is exactly the map induced on homology by the fiber inclusion $F \to E$. In particular, $H_p(F) \to H_p(E)$ is injective if and only if all differentials entering $E^r_{p,0}$ are trivial.*

2.2 Constructing the sseq of a filtered space

Up to now we have talked a lot about how to use the Serre spectral sequence, but we have yet to construct any spectral sequence. The reason for this is that spectral sequences are merely a tool for computation, so their construction doesn't really matter as long as we understand enough of their properties to use them. Nevertheless, it can be enlightening to have a look at the inner workings of a spectral sequence, so in this section we will construct the spectral sequence of a filtered space and then use it to construct the Leray–Serre spectral sequence.

Definition 2.2.1. A filtered space $X_\bullet$ consists of a space X and a sequence of subspaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots X.$$

Often one also allows X_k for $k < 0$, but this won't be relevant for us. We say that the filtration is exhaustive if $X = \bigcup_n X_n$ and X has the weak topology with respect to this union.

We will show the following:

Theorem 2.2.2. *For every exhaustively filtered space $X_\bullet$ there is a natural spectral sequence*

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \Rightarrow H_{p+q}(X).$$

Moreover, the d^1-differential is given by

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \xrightarrow{\partial} H_{p+q-1}(X_{p-1}) \longrightarrow H_{p+q-1}(X_{p-1}, X_{p-2}) = E^1_{p-1,q}.$$

Example 2.2.3. Consider the filtered space $\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \dots \subset \mathbb{CP}^n$. Applying the spectral sequence and using $\mathbb{CP}^k/\mathbb{CP}^{k-1} \simeq S^{2k}$ we get another way of computing the homology of $\mathbb{CP}^n$.

By applying the filtration spectral sequence to the cellular filtration of a CW complex we obtain a new proof of the following theorem.

Theorem 2.2.4. *For every CW complex X there is a canonical isomorphism*

$$H^{\text{cell}}_*(X) \cong H_*(X)$$

between the cellular and the singular homology of X.

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The spectral sequence of a filtered chain complex. Since spectral sequences are a beast of homological algebra, the basic input will be a chain complex rather than a space.

Definition 2.2.5. A filtered chain complex $F_\bullet C$ consists of a chain complex C and a sequence of subcomplexes

$$F_0 C \subseteq F_1 C \subseteq F_2 C \subseteq \dots C.$$

We say that the filtration is exhaustive if $C = \bigcup_n F_n C$.

Definition 2.2.6. A filtration on C induces a filtration on the homology: we write $F_n H_*(C)$ for the image of $H_*(F_n C) \to H_*(C)$.

Theorem 2.2.7. *For every filtered chain complex $F_\bullet C$ there is a natural spectral sequence with*

$$E^0_{p,q} = F_p C_{p+q}/F_{p-1}C_{p+q} \qquad\qquad E^1_{p,q} = H_{p+q}(E^0_{p,*}) = H_{p+q}(F_p C/F_{p-1}C)$$

and there are isomorphisms

$$E^\infty_{p,q} \cong \frac{F_p H_{p+q}(C)}{F_{p-1}H_{p+q}(C)}.$$

If the filtration in C was exhaustive, then so is the filtration on $H_*(C)$. This means that the E^∞-page of the spectral sequence is the associated graded of an exhaustive filtration on $H_*(C)$. In this case we say that the spectral sequence converges to $H_*(C)$ and we say that the spectral sequence has signature:

$$E^1_{p,q} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow H_*(C).$$

In order to construct the spectral sequence we have to make various homological algebra definitions.

Definition 2.2.8. We define the r-approximate cycles as

$$Z^r_{p,q} := \{c \in F_pC_{p+q} \mid d(c) \in F_{p-r}C_{p+q-1}\} = \ker(d\colon F_pC_{p+q} \longrightarrow F_pC_{p+q}/F_{p-r}C_{p+q-1})$$

and the r-approximate boundaries as

$$B^r_{p,q}* := \operatorname{Im}(d\colon F_{p+r+1}C_{p+q+1} \longrightarrow F_{p+r+1}C_{p+q}) \cap F_pC_{p+q}$$

so that we obtain a doubly-infinite sequence of subobjects:

$$0 = B^0_{p,q} \subseteq B^1_{p,q} \subseteq \ldots B^\infty_{p,q} \subseteq Z^\infty_{p,q} \subseteq \cdots \subseteq Z^1_{p,q} \subseteq Z^0_{p,q}$$

Now define the pages of the spectral sequence as

$$E^r_{p,q} := Z^r_{p,q}/(B^r_{p,q} + Z^{r-1}_{p-1,q+1})$$

and let the differential $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ be induced by the differential of the chain complex.

Lemma 2.2.9. *There are canonical isomorphisms*

$$H_*(E^r_p) \cong E^{r+1}_p$$

making this into a spectral sequence.

The Leray–Serre spectral sequence. We can now also construct the Leray–Serre spectral sequence. For simplicity we will work with $\mathbb{Z}$-coefficients throughout, thought the proof is the same with arbitrary coefficients. Let

$$F \longrightarrow E \longrightarrow B$$

be a fiber sequence where B is a CW complex. And define $E_n := p^{-1}(B^{(n)}) \subset E$ as the preimage of the n-skeleton of B. The spectral sequence of the filtered space $E_0 \subseteq \ldots E$ is of signature:

$$E^1_{p,q} = H_{p+q}(E_p, E_{p-1}) \Rightarrow H_{p+q}(E).$$

If $\pi_1(B)$ acts trivially on $H_*(F)$, then we can show the following:

Lemma 2.2.10. *If $\pi_1(B)$ acts trivially on $H_*(F)$, then the E^1-page of the above spectral sequence is isomorphic to the cellular chain complex of B with coefficients in $H_*(F)$:*

$$E^1_{p,q} = H_{p+q}(E_p, E_{p-1}) \cong C^{\text{cell}}_p(B) \otimes H_q(F) = C^{\text{cell}}_p(B; H_q(F))$$

and the d^1-differential is the cellular chain complex differential.

Since the E^2-page is the homology of the E^1-page with respect to the d^1-differential, it follows that

$$E^2_{p,q} \cong H^{\text{cell}}_p(B; H_q(F)).$$

So we have constructed the Leray–Serre spectral sequence.

2.3 The cohomology sseq and multiplicative structures

Cohomological Serre grading.

Definition 2.3.1. A spectral sequence $(E_{\bullet,\bullet}, d)$ in cohomological Serre grading, starting on page $r_0 \geq 1$, consists of:

1. for each $r \geq r_0$ a bigraded abelian group $(E^r_{p,q})_{p,q\in\mathbb{Z}}$.
2. For all $r \geq r_0$ and $p, q \in \mathbb{Z}$ a map of abelian groups

$$d_r^{p,q} \colon E^r_{p,q} \longrightarrow E^r_{p+r,q-r+1}$$

satisfying $d_r^{p,q} \circ d_r^{p+r,q+r-1} = 0$.

3. For all $r \geq r_0$ and $p, q \in \mathbb{Z}$, isomorphisms of abelian groups

$$E^{r+1}_{p,q} \cong \frac{\ker\left(d_r^{p,q} \colon E_r^{p,q} \longrightarrow E_r^{p+r,q+r-1}\right)}{\operatorname{im}\left(d_r^{p-r,q-r+1} \colon E_r^{p-r,q-r+1} \longrightarrow E_r^{p,q}\right)}$$

For simplicity these two groups are usually assumed to be equal, rather than just isomorphic.

There is a Leray–Serre spectral sequence for cohomology, which is entirely analogous to the one for homology.

Theorem 2.3.2. *For every abelian group G and every fiber sequence*

$$F \longrightarrow E \longrightarrow B$$

such that $\pi_1(B)$ acts trivially on $H^*(F; G)$ there is a (natural, convergent) Leray–Serre spectral sequence in cohomological Serre grading of signature

$$E_2^{p,q} = H^p(B; H^q(F; G)) \Rightarrow H^{p+q}(E; G)$$

meaning that the E_2-page is given by $E_2^{p,q} = H_p(B; H_q(F; G))$ and there is a natural filtration

$$H^n(E; G) = F_0^n \supseteq F_1^n \supseteq \cdots \supseteq F_n^n \supseteq F_{n+1}^n = 0$$

and natural short exact sequences:

$$0 \to F_{p+1}^{p+q} \hookrightarrow F_p^{p+q} \twoheadrightarrow E^\infty_{p,q} \to 0$$

Example 2.3.3. Consider again the following fiber sequence from Example 2.1.10

$$S^1 \longrightarrow \mathbb{RP}^3 \xrightarrow{\overline{\eta}} \mathbb{CP}^1$$

The second page of the cohomology spectral sequence looks as follows:

$$\begin{tikzcd}
1 & \mathbb{Z} \arrow[drr, "\cdot 2"'] & & \mathbb{Z} \\
0 & \mathbb{Z} & & \mathbb{Z} \\
 & 0 & 1 & 2
\end{tikzcd}$$

Now there must be a d_2 differential, and it must be given by multiplication by 2 (or by -2), since we know that $H^2(\mathbb{RP}^3) = \mathbb{Z}/2$. This results in the following E^3-page:

$$\begin{array}{c|ccc}
1 & 0 & & \mathbb{Z} \\
0 & \mathbb{Z} & & \mathbb{Z}/2 \\
\hline
 & 0 & 1 & 2
\end{array}$$

As usual we can read off the homology of the total space by looking at anti-diagonals. Note that in contrast to Example 2.1.10 the group $\mathbb{Z}/2$ appears in degree 2 rather than in degree 1, and this is compatible with the universal coefficient theorem.

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The multiplicative structure. The cohomological Leray–Serre spectral sequence interacts well with the cup product on cohomology. Concretely we have the following:

Theorem 2.3.4. *Suppose that R is a ring and $F \to E \to B$ is a fiber sequence such that $\pi_1(B)$ acts trivially on $H^*(F;R)$. Then there is are multiplication maps*

$$- \cdot -\colon E_r^{p,q} \otimes E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$

such that

1. *$E_r^{\bullet,\bullet}$ is a bigraded commutative ring for all $r \geq 2$.*
2. *d_r is a derivation in the sense that*
$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{|x|} x \cdot d_r(y)$$
where the degree of $x \in E_r^{p,q}$ is $|x| = p + q$.
3. *The isomorphism $E_{r+1}^{\bullet,\bullet} \cong H_*(E_r^{\bullet,\bullet})$ is compatible with the product.*
4. *The isomorphism $E_2^{p,q} \cong H_p(B; H_q(F))$ is compatible with the product.*
5. *The filtration $F_\bullet^n$ satisfies that $F_a^n \cdot F_b^m \subseteq F_{a+b}^{n+m}$ with respect to the cup product on $H^*(E)$.*[4]

[4]Note that this is a property of the filtration and of the cup product on $H^*(E)$, not a property of the multiplication on the spectral sequence.

6. *The isomorphisms* $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$ *are compatible with the ring structures.*[5]

Example 2.3.5. In the past we have used the homotopy fiber sequence

$$K(\mathbb{Z},1) \longrightarrow * \longrightarrow K(\mathbb{Z},2)$$

to compute that $H_*(K(\mathbb{Z},2)) \cong \mathbb{Z}$ for $*$ even and 0 otherwise. We can now use the multiplicative structure of the spectral sequence to determine the ring structure. Let $c_n \in H_{2n}(K(\mathbb{Z},2))$ be a generator and write $x \in H_1(S^1)$ for the canonical generator. Then we must have $d_2(x \cdot c_n) = \pm c_{n+1}$, but by the Leibniz rule we also have:

$$d_2(x \cdot c_n) = d_2(x) \cdot c_n + (-1)^{|x|} x \cdot d_2(c_n) = c_1 \cdot c_n$$

So we conclude that $c_{n+1} = \pm c_1 \cdot c_n$, and up to rechoosing generators we may assume that the sign is +. This shows that $H^*(K(\mathbb{Z},2)) \cong \mathbb{Z}[c]$, which we had previously deduced using Poincaré duality for $\mathbb{CP}^n$.

Motivated by the previous example, we take on the cohomology of ΩS^n again.

Example 2.3.6. We know that $H^{k(n-1)}(\Omega S^n) \cong \mathbb{Z}\langle x_k \rangle$ and 0 if the degree is not divisible by $(n-1)$. Let $y \in H^n(S^n)$ be the canonical generator. Then we know that $d_{n-1}(x_k) = x_{k-1} \cdot y$ (in principle there could be a minus sign, but we can rechoose x_k to make it disappear). On the other hand, the Leibniz rule tells us that:

$$\begin{aligned} d_{n-1}(x_k \cdot x_l) &= d_{n-1}(x_k) \cdot x_l + (-1)^{|x_k|} x_k \cdot d_{n-1}(x_l) = x_{k-1} \cdot y \cdot x_l + (-1)^{k(n-1)} x_k \cdot x_{l-1} \cdot y \\ &= (x_{k-1} \cdot x_l + (-1)^{k(n-1)} x_k \cdot x_{l-1}) \cdot y \end{aligned}$$

There are integers $a_{k,l}$ such that $x_k \cdot x_l = a_{k,l} x_{k+l}$. Inserting these we see:

$$d_{n-1}(a_{k,l} x_{k+l}) = (a_{k-1,l} + (-1)^{k(n-1)} a_{k,l-1}) x_{k+l-1} \cdot y$$

which implies the recursion relation Therefore $a_{k,l} = a_{k-1,l} + (-1)^{k(n-1)} a_{k,l-1}$. Let us assume that n is odd so that the sign disappears. (The case where n is even is slightly more complicated, see [Ran21, p. 53].) Then the only solution to these equations are the binomial coefficients and we get:

$$x_k \cdot x_l = \binom{k+l}{k} x_{k+l}$$

This means that $H^*(\Omega S^n)$ is the "free divided power algebra on $x \in H^{n-1}$", i.e. it is isomorphic to

$$\mathbb{Z}[x, \tfrac{x^2}{2!}, \tfrac{x^3}{3!}, \tfrac{x^4}{4!}, \dots] \subset \mathbb{Q}[x]$$

where $x_k = \frac{x^k}{k!}$.

[5] The right hand side is the (p,q)-piece of a well-defined bigraded ring by the previous item.

The first stable homotopy group. Our goal is now to use the new tool we have to compute $\pi_{n+1}(S^n)$. By the Freudenthal suspension theorem it will suffice to do so for $n = 3$.

In the previous subsection we saw that the cohomology of $K(\mathbb{Z}, 2)$ looks a lot like the cohomology of ΩS^3, except for its ring structure. Note that if the two rings were indeed the same, one could show that $\Omega S^3 \simeq K(\mathbb{Z}, 2)$, which would imply that S^3 is a $K(\mathbb{Z}, 3)$ and hence does not have any higher homotopy groups. Now that we know that $\Omega S^3 \not\simeq K(\mathbb{Z}, 2)$ we can try to leverage the failure of this equivalence to compute the first non-trivial homotopy group $\pi_4(S^3)$.

Example 2.3.7. From the Whitehead tower we get a fiber sequence

$$K(\mathbb{Z}, 2) \longrightarrow \tau_{>3}S^3 \longrightarrow S^3$$

where $\pi_k(\tau_{>3}S^3) \cong \pi_k(S^3)$ for $k > 3$ and trivial otherwise. The cohomological LS-sseq looks as follows:

$$\begin{tikzcd}[row sep=small]
4 & \mathbb{Z}\langle c^2\rangle \arrow[rrrdd, "\cdot 2"] & & & \mathbb{Z}\langle c^2x\rangle \\
3 & & & & \\
2 & \mathbb{Z}\langle c\rangle \arrow[rrrdd, "\cong"] & & & \mathbb{Z}\langle cx\rangle \\
1 & & & & \\
0 & \mathbb{Z}\langle 1\rangle & & & \mathbb{Z}\langle x\rangle \\
 & 0 & 1 & 2 & 3
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
4 & 0 & & & \mathbb{Z}/3\langle c^2x\rangle \\
3 & & & & \\
2 & 0 & & & \mathbb{Z}/2\langle cx\rangle \\
1 & & & & \\
0 & \mathbb{Z}\langle 1\rangle & & & 0 \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

We conclude that $H^5(\tau_{>3}S^3) = \mathbb{Z}/2$ and hence $\pi_4(\tau_{>4}S^3) = H_4(\tau_{>3}S^3) = \mathbb{Z}/2$.

This example, together with the Freudenthal suspension theorem shows:

Theorem 2.3.8. *For $n \geq 3$ we have $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$. Moreover, the non-zero class $S^{n+1} \to S^n$ can be represented by the $(n-1)$-fold suspension of the Hopf map.*

This indicates a general strategy for computing homotopy groups of spheres (or really of any simply connected space X). Suppose we know the cohomology of X and the cohomology of $K(G, n)$ for any finitely generated abelian group G. Then we can inductively (try to) use the cohomology LS-sseq of the fiber sequence

$$K(\pi_n X, n-1) \longrightarrow \tau_{>n}X \longrightarrow \tau_{>n-1}X$$

to compute the cohomology of $\tau_{>n}X$, and from it read off the lowest degree homology group, which by Hurewicz is $H_{n+1}(\tau_{>n}X) \cong \pi_{n+1}(\tau_{>n}X) \cong \pi_{n+1}(X)$.

Example 2.3.9. To compute $\pi_5(S^3)$ we would now consider the fiber sequence

$$K(\mathbb{Z}/2, 3) \longrightarrow \tau_{>4}S^3 \longrightarrow \tau_{>3}X.$$

However, we don't know the cohomology of $K(\mathbb{Z}/2, 3)$, so we can't even attempt to run the spectral sequence.

Cohomology of EM-spaces, first attempt. To have any hope of following through on the aforementioned strategy, we need to know the cohomology of $K(G, n)$ for finitely generated abelian groups G.

Remark 2.3.10. We have $K(G \times H, n) \simeq K(G, n) \times K(H, n)$. Therefore it suffices to understand $K(\mathbb{Z}, n)$ and $K(\mathbb{Z}/p^r, n)$ for p prime.

Example 2.3.11. To compute the cohomology of $K(\mathbb{Z}, 3)$ we can use the homotopy fiber sequence

$$K(\mathbb{Z}, 2) \longrightarrow * \longrightarrow K(\mathbb{Z}, 3)$$

Up to dimension 13 the cohomology ring of $K(\mathbb{Z}, 3)$ is isomorphic to the graded polynomial algebra

$$\mathbb{Z}[x_3, y_8, z_{10}, w_{12}]/\langle 2x_3^2, 3y_8, 2z_{10}, 5w_{12}\rangle$$

where $|x_3| = 3$, $|y_8| = 8$, $|z_{10}| = 10$, $|w_{12}| = 12$. However, in dimension 13 it difficult to work out what happens as there is potential for some longer differentials that we cannot work out using the multiplicative structure on the E_3-page.

More generally we have that $\Omega K(\mathbb{Z}, n) = K(\mathbb{Z}, n-1)$, and so there is a homotopy fiber sequence:

$$K(\mathbb{Z}, n-1) \longrightarrow * \longrightarrow K(\mathbb{Z}, n)$$

In general applying the cancellation argument here is quite hard, so let us only consider homology with $\mathbb{Q}$-coefficients for now.

Lemma 2.3.12. *The rational homology of the integral Eilenberg–Mac Lane spaces is given by:*

$$H_*(K(\mathbb{Z}, 2n+1); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & * = 0, 2n+1 \\ 0 & \textit{otherwise,} \end{cases} \qquad H_*(K(\mathbb{Z}, 2n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & 2n \textit{ divides } *, \\ 0 & \textit{otherwise.} \end{cases}$$

Multiplicative extensions. There is a subtlety regarding multiplicative spectral sequences that we have not yet touched on, as it only becomes relevant if we want to compute the ring structure on $H^*(E)$. (So far we have mostly used this spectral sequence to compute $H^*(F)$ or $H^*(B)$.)

Example 2.3.13. Consider the fiber sequence

$$S^2 \longrightarrow \mathbb{CP}^3 \longrightarrow S^4$$

obtained by taking the quotient of the total space of the quaternionic Hopf fibration $S^7 \to S^4$ by the U_1-action. The spectral sequence looks as follows

2	$\mathbb{Z}\langle x\rangle$				$\mathbb{Z}\langle xy\rangle$
1					
0	$\mathbb{Z}\langle 1\rangle$				$\mathbb{Z}\langle y\rangle$
	0	1	2	3	4

We can see that there is no room for differentials and so the spectral sequence collapses. At first sight this might suggest that

$$H^*(E) \cong \Lambda_{\mathbb{Z}}[x, y]$$

with $|x| = 2$ and $|y| = 4$. But we know that in $H^*(\mathbb{CP}^3)$ we have $x^2 = y$ instead of $x^2 = 0$. This is known as a multiplicative extension. The reason that this can happen is that Theorem 2.3.4 only guarantees us an isomorphism of graded rings

$$\bigoplus_{p,q} E_\infty^{p,q} \cong \bigoplus_{p,q} \frac{F_p^{p+q}}{F_{p+1}^{p+q}}$$

for some multiplicative filtration $F_\bullet^n \subseteq H^n(E)$. In the example at hand the filtration on $H^*(\mathbb{CP}^3) = \mathbb{Z}[c]/c^4$ is:

$$F_0^* = \mathbb{Z}\langle 1, c, c^2, c^3\rangle \supseteq F_1^* = F_2^* = F_3^* = F_4^* = \mathbb{Z}\langle c^2, c^3\rangle \supseteq F_5^* = 0$$

and hence its associated graded is (note there here the quotients are of abelian groups not of rings)

$$\bigoplus_{p\geq 0} \frac{F_p^*}{F_{p+1}^*} \cong \frac{F_0^*}{F_1^*} \oplus \frac{F_4^*}{F_5^*} \cong \mathbb{Z}[c]/c^2 \oplus c^2 \cdot \mathbb{Z}[c]/c^2$$

Writing $x := c$ and $y := c^2$ we see that this associated graded ring is indeed $\Lambda_{\mathbb{Z}}[x, y]$.

Example 2.3.14. Another instance of multiplicative extensions arises in the Leray–Serre spectral sequence with $\mathbb{F}_2$-coefficients for the fiber sequence

$$S^1 \longrightarrow K(\mathbb{Z}/2, 1) \longrightarrow K(\mathbb{Z}, 2).$$

We know that $K(\mathbb{Z}/2, 1) \simeq \mathbb{RP}^\infty$, so $H^*(K(\mathbb{Z}/2, 1); \mathbb{F}_2) \cong \mathbb{F}_2[x]$ with $|x| = 2$. But the $E_2 = E_\infty$-page is

1	$\mathbb{F}_2\langle x\rangle$		$\mathbb{F}_2\langle x\alpha\rangle$		$\mathbb{F}_2\langle x\alpha^2\rangle$
0	$\mathbb{F}_2\langle 1\rangle$		$\mathbb{F}_2\langle \alpha\rangle$		$\mathbb{F}_2\langle \alpha^2\rangle$
	0	1	2	3	4

which as a graded ring is $\Lambda_{\mathbb{F}_2}\langle x\rangle \otimes \mathbb{F}_2[\alpha]$. However, this just means that $x^2 \equiv 0$ modulo higher filtration degree. In the cohomology of the total space we then have that $x^2 = \alpha$, which is allowed because α has fitlration degree 2.

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2.4 Serre classes and the mod $\mathcal{C}$ Hurewicz theorem

We have seen that (in a range) $K(\mathbb{Z}/p, 2)$ only has homology that is p^r-torsion for some r. This suggests a general principle by which a space with p-power-torsion homotopy groups should have p-power-torsion homology groups and vice-versa.

Serre classes.

Definition 2.4.1. A Serre class $\mathcal{C}$ is a collection of isomorphism classes of abelian groups such that for any short exact sequence:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have that $B \in \mathcal{C}$ if and only if A and $C \in \mathcal{C}$.

Example 2.4.2. Here are the most important examples of Serre classes:

1. $\mathscr{F}\mathscr{G}$ the class of finitely generated abelian groups is a Serre class.
2. For any subset P of the set of primes, let $\mathscr{T}_P$ be the class of torsion abelian groups, where each element has order only divisible by primes in P. This is a Serre class.
3. Let $\mathcal{F}_P \subset \mathcal{T}_P$ be the subclass of those groups that are moreover finite. (Equivalently this is the class of those finite abelian groups whose order is a product of primes in P.) This is a Serre class.
4. The class of torsion-free abelian groups is not a Serre class.
5. The class of abelian groups of that are 9-torsion is not a Serre class.

Definition 2.4.3. We say that a morphism of abelian groups $f\colon A \to B$ is

1. an epi mod $\mathcal{C}$ if $\operatorname{coker}(f) \in \mathcal{C}$,
2. a mono mod $\mathcal{C}$ if $\ker(f) \in \mathcal{C}$,
3. an iso mod $\mathcal{C}$ if it is both and epi and a mono mod $\mathcal{C}$.

Remark 2.4.4. A map $f\colon A \to B$ is an iso mod $\mathcal{T}_P$ if and only if

$$A \otimes \mathbb{Z}[P^{-1}] \longrightarrow B \otimes \mathbb{Z}[P^{-1}]$$

is an iso. In particular, if P is the set of all primes, then f is an iso mod $\mathcal{T}_P$ if and only if $f \otimes \mathrm{id}_{\mathbb{Q}}$ is an iso. If P is all primes but p, then f is an iso mod $\mathcal{T}_P$ if and only if $f \otimes \mathrm{id}_{\mathbb{Z}_{(p)}}$ is an iso.

Lemma 2.4.5. *Given two morphisms $f\colon A \to B$ and $g\colon B \to C$, if two of the three morphisms f, g, $g \circ f$ is an iso mod $\mathcal{C}$, then so is the third.*

Definition 2.4.6. We say that a Serre class $\mathcal{C}$ is good if moreover satisfies the following two conditions:

(a) for all $A, B \in \mathcal{C}$ we also have $A \otimes B, \operatorname{Tor}(A, B) \in \mathcal{C}$.

(b) for all $A \in \mathcal{C}$ we have $H_k(K(A, 1); \mathbb{Z}) \in \mathcal{C}$ for all $k \geq 1$.

The main use of good Serre classes is the following:

Proposition 2.4.7. *Let $\mathcal{C}$ be a good Serre class. Suppose $F \to E \to B$ is a fiber sequence (with connected base) such that $\pi_1(B)$ acts trivially on $H_*(F; \mathbb{Z})$. Then if two of the groups*

$$\widetilde{H}_*(F; \mathbb{Z}), \quad \widetilde{H}_*(E; \mathbb{Z}), \quad \widetilde{H}_*(B; \mathbb{Z}),$$

are in $\mathcal{C}$, then so is the third.

Lemma 2.4.8. *All of $\mathscr{F}\mathscr{G}$, $\mathcal{T}_P$, and $\mathcal{F}_P$ are good Serre classes.*

Lemma 2.4.9. *If $\mathcal{C}$ is a good Serre class, then $H_k(K(A, n); \mathbb{Z}) \in \mathcal{C}$ for all $k, n \geq 1$ and $A \in \mathcal{C}$.*

Mod $\mathcal{C}$ Hurewicz. We can now prove a Hurewicz theorem mod $\mathcal{C}$.

Theorem 2.4.10. *Let $\mathcal{C}$ be a good Serre class. Let X be a simply connected space with $\pi_i(X) \in \mathcal{C}$ for $0 < i < n$. Then $H_i(X) \in \mathcal{C}$ for $0 < i < n$ and the Hurewicz homomorphism*

$$\hbar\colon \pi_n(X) \longrightarrow H_n(X)$$

is an iso mod $\mathcal{C}$.

Corollary 2.4.11. *Suppose that X is a simply connected space such that $H_k(X)$ is finitely generated for all k. Then $\pi_k(X)$ is finitely generated for all k. In particular, the group $\pi_i(S^n)$ is finitely generated for all $i, n \in \mathbb{N}$.*

Corollary 2.4.12. *The groups $\pi_k(S^3)$ are finite for all $k > 3$.*

Corollary 2.4.13. *The first p-torsion in $\pi_k(S^3)$ appears for $k = 2p$ and we have that $\pi_k(S^3) \cong \mathbb{Z}/p \oplus A$ where A is a finite abelian group of order coprime to p.*

Theorem 2.4.14. *The homotopy groups of spheres satisfy*

$$\pi_k(S^n) \cong \begin{cases} 0 & \text{for } k < n, \\ \mathbb{Z} & \text{for } k = n, \\ \mathbb{Z} \oplus F_{n,k} & \text{for } k = 2n-1, \text{ and } n \text{ even}, \\ F_{n,k} & \text{otherwise}. \end{cases}$$

where $F_{n,k}$ is some finite abelian group.

[The following wasn't discussed in the lectures]

Theorem 2.4.15. *After stabilising the Hurewicz homomorphism becomes a rational isomorphism:*

$$\hbar\colon \pi_*^{\mathrm{st}}(X) \otimes \mathbb{Q} \xrightarrow{\cong} H_*(X;\mathbb{Q}).$$

Here $\pi_k^{\mathrm{st}}(X) := \operatorname{colim}_{n\to\infty} \pi_{k+n}(\Sigma^n X)$, which may also be computed as $\pi_{2n+2}(\Sigma^{n+2}X)$ because of the Freudenthal suspension theorem.

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2.5 Steenrod operations and the cohomology of EM-spaces

Axioms for Steenrod operations.

Definition 2.5.1. The Steenrod operations are the unique maps

$$\mathrm{Sq}^i\colon H^n(X;\mathbb{F}_2) \longrightarrow H^{n+i}(X;\mathbb{F}_2)$$

for all spaces X and all $i, n \geq 0$ satisfying the following axioms:

1. $\mathrm{Sq}^i(f^*a) = f^*\mathrm{Sq}^i(a)$ for all $f\colon X \to Y$, (the Sq^i are natural),

2. $\mathrm{Sq}^i(a+b) = \mathrm{Sq}^i(a+b)$ (the Sq^i are additive),

3. $\mathrm{Sq}^i(a \cup b) = \sum_{j+k=i} \mathrm{Sq}^j(a) \cup \mathrm{Sq}^k(b)$ (the Cartan formula),

4. $\mathrm{Sq}^i(\sigma(a)) = \sigma(\mathrm{Sq}^i(a))$ where $\sigma\colon H^n(X;\mathbb{F}_2) \cong H^{n+1}(X;\mathbb{F}_2)$ is the suspension isomorphism, (the Sq^i commute with suspension),

5. $\mathrm{Sq}^i(a) = a^2$ for $|a| = i$ and $\mathrm{Sq}^i(b) = 0$ for $|b| < i$, (the Sq^i square elements of degree i),

6. $\mathrm{Sq}^0(a) = a$,

7. $\mathrm{Sq}^1(a) = \beta(a)$ where β is the Bockstein homomorphism coming from the coefficient short exact sequence $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$.

We will not prove the following theorem in the lectures, though it is possible to do so using the techniques we have discussed. See for instance [Ran21, Chapter 3].

Theorem 2.5.2. *The Steenrod operations exist and are unique.*

Example 2.5.3. We know that the mod 2 cohomology ring of $\mathbb{CP}^2$ is

$$H^*(\mathbb{CP}^2;\mathbb{F}_2) \cong \mathbb{F}_2[c]/c^3.$$

Since $|c| = 2$ we have that $\mathrm{Sq}^2(c) = c^2$ and of course $\mathrm{Sq}^0(c) = 0$. All other $\mathrm{Sq}^i(c)$ are 0. Now consider the suspension $\Sigma\mathbb{CP}^2$. Its cohomology ring is

$$H^*(\Sigma\mathbb{CP}^2;\mathbb{F}_2) \cong \mathbb{F}\langle 1, x, y\rangle$$

where $|x| = 3$ and $|y| = 5$. The ring structure is necessarily trivial, but by axiom 5 we know that $\mathrm{Sq}^2(x) = y$. (Because $x = \sigma(c)$ and $y = \sigma(c^2)$.

If we, on the other hand, compute $H^*(S^3 \vee S^5;\mathbb{F}_2) \cong \mathbb{F}\langle 1, x, y\rangle$, then $\mathrm{Sq}^2(x) = 0$ by naturality. Therefore, we can use the Steenrod squares to show that $\Sigma\mathbb{CP}^2 \not\simeq S^3 \vee S^5$. In fact, the same argument shows that $\Sigma^n\mathbb{CP}^2 \not\simeq S^{n+1} \vee S^{n+3}$.

Remark 2.5.4. A good way to think about the Steenrod operations is via the total Steenrod square:

$$\begin{aligned} \mathrm{Sq}\colon H^*(X;\mathbb{F}_2) &\longrightarrow H^*(X;\mathbb{F}_2) \\ a &\longmapsto \sum_{i \geq 0} \mathrm{Sq}^i(a) \end{aligned}$$

The axioms can now be summarised as follows:

1+4. Sq is natural and commutes with suspension.

2+3. Sq is a ring homomorphism.

5+6+7. For $a \in H^n(X;\mathbb{F}_2)$, $\mathrm{Sq}(a)$ is of the form

$$\mathrm{Sq}(a) = a + \beta(a) + \mathrm{Sq}^2(a) + \cdots + \mathrm{Sq}^{n-1}(a) + a^2.$$

Example 2.5.5. We know that the mod 2 cohomology ring of $\mathbb{RP}^\infty$ is

$$H^*(\mathbb{RP}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x].$$

Moreover, $\mathrm{Sq}(x) = x + x^2 = x(x+1)$ since $|x| = 1$. So it follows that

$$\mathrm{Sq}(x^n) = (\mathrm{Sq}(x))^n = x^n(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n+i}$$

Remark 2.5.6. The Bockstein homomorphism $\beta = \mathrm{Sq}^1 \colon H^*(X;\mathbb{F}_2) \to H^*(X;\mathbb{F}_2)$ can be used to tell when two copies of $\mathbb{F}_2$ come from the same copy of a $\mathbb{Z}/2$ in homology. Concretely, if we have $H_i(X;\mathbb{Z}) \cong \mathbb{Z}/2$, then in cohomology with $\mathbb{F}_2$-coefficients there will be a class $x \in H^i(X;\mathbb{F}_2)$ and $x' \in H^{i+1}(X;\mathbb{F}_2)$. The Bockstein will be $\mathrm{Sq}^1(x) = x'$. If however, $H_i(X;\mathbb{Z}) \cong \mathbb{Z}$, then there is a single corresponding class $x \in H^i(X;\mathbb{F}_2)$ and $\mathrm{Sq}^1(x) = 0$. (If $H_i(X;\mathbb{Z}) \cong \mathbb{Z}/2^r$ with $r > 1$ then there will be two classes on which the Bockstein is trivial.)

Transgressive pairs. For a fiber sequence $F \to E \to B$ consider the commutative diagram:

$$\begin{tikzcd}
H^i(F) \arrow[r, "\delta"] & H^{i+1}(E,F) & \\
& H^{i+1}(B,b_0) \arrow[u, "\pi^*"] \arrow[r, "\cong"] & H^{i+1}(B).
\end{tikzcd}$$

Definition 2.5.7. We say that $(x,y) \in H^i(F) \times H^{i+1}(B)$ is a transgressive pair if $\delta(x) = \pi^*\widetilde{y}$, where $\widetilde{y} \in H^{i+1}(B,b_0)$ is the unique class corresponding to $y \in H^{i+1}(B)$.

Remark 2.5.8. If in the above diagram we write π_i instead of H^i (and we turn around the arrows) then the vertical map is an isomorphism and this is exactly the definition of the boundary map in the π_* LES of a fiber sequence.

We can think of transgressive pairs as defining a multi-valued and partially defined map:

$$H^i(F) \overset{\delta}{\dashrightarrow} H^{i+1}(B).$$

These transgressive pairs turn up as "long differentials" in the Leray–Serre spectral sequence:

Lemma 2.5.9. *A pair $(x,y) \in H^i(F) \times H^{i+1}(B)$ is transgressive if and only if:*

1. *$x \in H^i(F) = E_2^{0,i}$ "survives" to the E_{i+1}-page, i.e. $0 = d_2x = d_3x = \cdots = d_{i-1}x$, and*
2. *$d_i(x) = [y] \in E_i^{i+1,0}$.*

Here we note that $E_i^{i+1,0}$ is a quotient of $E_2^{i+1,0} = H^{i+1}(B)$, so y induces a well-defined element.

Therefore, if we have a way of producing transgressive pairs, then this will help us to determine differentials. The main source of transgressive pairs are the Steenrod operations. In fact, it follows from the definition the Steenrod operations preserve transgressive pairs:

Lemma 2.5.10. *If $(x,y) \in H^i(F;\mathbb{F}_2) \times H^{i+1}(B;\mathbb{F}_2)$ is a transgressive pair, then so is $(\mathrm{Sq}^j(x), \mathrm{Sq}^j(y))$ for all $j \geq 0$.*

Borel's theorem.

Definition 2.5.11. We say that a sequence of elements $x_1, x_2, \dots \in H^*(X;\mathbb{F})$ is a multiplicative basis if the elements

$$\{x_{i_1} \cdot \dots \cdot x_{i_k} \mid i_1 < \dots < i_k\}$$

form an (additive) basis of the vector space $H^*(X;\mathbb{F})$.

Example 2.5.12. The cohomology ring $H^*(S^3 \times S^5 \times S^{11}) \cong \Lambda_{\mathbb{F}}\langle x_3, x_5, x_{11}\rangle$ has a multiplicative basis given by x_3, x_5, and x_{11}. The cohomology ring $H^*(\mathbb{CP}^\infty;\mathbb{F}) \cong \mathbb{F}[c]$ has a multiplicative basis given by c, c^2, c^4, c^8, ...

Theorem 2.5.13 (Borel)**.** *Suppose we have a fiber sequence $F \to X \to B$ with X contractible, B simply connected, and there exists a sequence $(x_1, y_1), (x_2, y_2), \dots$ of transgressive pairs such that the $x_1, x_2, \dots$ form a multiplicative basis of $H^*(F;\mathbb{F})$. Assume further that either $\mathbb{F}$ has charactistic 2, or all x_i are of odd degree. Then the cohomology of B is a polynomial algebra on the y_i:*

$$H^*(B;\mathbb{F}) \cong \mathbb{F}[y_1, y_2, \dots]$$

***The spectral sequence comparison theorem.**

The cohomology of Eilenberg–Mac Lane spaces.

Definition 2.5.14. For $I = (i_1, \dots, i_n)$ a sequence of natural numbers we define $\mathrm{Sq}^I := \mathrm{Sq}^{i_1} \dots \mathrm{Sq}^{i_n}$ as the composite of Steenrod squares. We say that I is admissible if $i_k \geq 2i_{k+1}$ for all k and we define the excess of I as

$$e(I) := \sum_{k=1}^{n} (i_k - 2i_{k+1}) = i_1 - (i_2 + i_3 + \dots + i_n)$$

where by convention we set $i_{n+1} = 0$.

Recall that there is a "canonical class" $\iota_n \in H^n(K(\mathbb{Z}/2, n), \mathbb{F}_2)$ such that[6]

$$\begin{aligned} [X, K(\mathbb{Z}/2, n)] &\longrightarrow H^n(X;\mathbb{F}_2) \\ [f] &\longmapsto f^*\iota_n \end{aligned}$$

is an isomorphism for any CW complex X. We can apply Steenrod operations to this class to obtain new elements in the cohomology of $K(\mathbb{Z}/2, n)$:

$$\mathrm{Sq}^I \iota_n \in H^{n+i_1+\dots+i_k}(K(\mathbb{Z}/2, n), \mathbb{F}_2)$$

Remark 2.5.15. If I is admissible and of excess $e(I)$, then we have $\mathrm{Sq}^I x = 0$ for any class x of degree $|x| < e(I)$. In particular $\mathrm{Sq}^I \iota_n = 0$ whenever $n < e(I)$.

[6]Here $\mathbb{Z}/2$ and $\mathbb{F}_2$ denote the same thing. I write $\mathbb{Z}/2$ when I mean the abelian group and $\mathbb{F}_2$ when I mean the field.

Remark 2.5.16. There are certain relations between Steenrod operations, called the Adem relations. For example it is true that $\mathrm{Sq}^1\mathrm{Sq}^{2i} = \mathrm{Sq}^{2i+1}$. In general the Adem relation says that for $0 < i < 2j$ we have

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{i/2} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k.$$

We will not prove these relations in this course, though we might use some of them later. A proof can be found in [Hat02, Theorem 4L.16]. One consequence of the Adem relations is that for every non-admissible I the operation Sq^I can be rewritten in terms of admissible operations. We will see another proof of this now, which doesn't use the Adem relations.

Theorem 2.5.17 (Serre)**.** *The mod 2 cohomology ring of $K(\mathbb{Z}/2, n)$ is a polynomial algebra on the admissible Steenrod operations of excess $< n$:*

$$H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2) \cong \mathbb{F}_2[\mathrm{Sq}^I \iota_n : I = (i_1, \ldots, i_n) \text{ admissible of excess } e(I) < n]$$

Corollary 2.5.18. *For $n = 1, 2$ we have*

$$H^*(K(\mathbb{Z}/2, 1); \mathbb{F}_2) \cong \mathbb{F}_2[\iota_1]$$
$$H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \cong \mathbb{F}_2[\iota_2, \mathrm{Sq}^1\iota_2, \mathrm{Sq}^{(2,1)}\iota_2, \mathrm{Sq}^{(4,2,1)}\iota_2, \ldots]$$

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2.6 Some homotopy groups of spheres

The goal of this section is to combine what we've discussed so far to compute $\pi_k(S^n)$ for some small values of k and n. In particular we will show:

Theorem 2.6.1. *The first few higher homotopy groups of S^3 are:*

$$\pi_4(S^3) \cong \mathbb{Z}/2 \qquad \pi_5(S^3) \cong \mathbb{Z}/2 \qquad \pi_6(S^3) \cong \mathbb{Z}/12.$$

Remark 2.6.2. While we have previously used the Whitehead tower to compute the first few homotopy groups of spaces, it turns out that in the case of spheres it is more effective to use the Postnikov tower. The reason for this is that $H^*(\tau_{>3}S^3)$ doesn't seem to have an interesting ring structure or action of the Steenrod operations, making it hard to control the differentials when trying to compute $H^*(\tau_{>4}S^3)$.

Recall that the Postnikov tower of S^n starts with $\tau_{\leq n}S^n = K(\mathbb{Z}, n)$ and that there are fiber sequences

$$K(\pi_k(S^n), k) \longrightarrow \tau_{\leq k}S^n \longrightarrow \tau_{\leq k-1}S^n.$$

We will inductively compute the cohomology of $\tau_{\leq k-1}S^n$. This might seems slightly tautological, as we need to know $\pi_k(S^n)$ to run the Leray–Serre spectral sequence for the above fiber sequence, but it turns out that we can already read off $\pi_k(S^n)$ from the cohomology of $\tau_{\leq k-1}$. (By thinking about what the spectral sequence must look like and using that $H_*(\tau_{\leq k-1}S^n) \cong H_*(S^n)$ for $* \leq k$.)

Lemma 2.6.3. *For $k > n$ there are isomorphisms*

$$\pi_k(S^n) \cong H_{k+1}(\tau_{\leq k-1}S^n) \cong H^{k+2}(\tau_{\leq k-1}S^n)$$

where the second isomorphism comes from the UCT and isn't natural.

As we already know that the first 3-torsion in $\pi_*(S^3)$ is in dimension 6 and is of the form

$$\pi_6(S^3) \cong \mathbb{Z}/3 \oplus A,$$

where A is an abelian group of order 2^r, it will suffice to compute the 2-power torsion in $\pi_*(S^3)$ for $* \leq 6$. So it will suffice to compute

$$H^*(\tau_{\leq k}S^3; \mathbb{Z}_{(2)})$$

for $k = 3, 4, 5$ and $* \leq 8$.

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